Crank-Nicolson method for the solution of linear convection-diffusion equation in two dinmnsions

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***Abstract*— Numerical methods for solving problems with differential equations are generally based on the local approximate representation of the solution using elementary functions, usually polynomials. Historically, and depending on the mathematical tools used, two main methods are distinguished: the finite difference method and the finite element method. The finite difference method is based on the local replacement of derivatives with differences. In this work, we provide a general overview of the analytical and numerical solutions of the two-dimensional linear convection-diffusion equation, using finite difference methods. The well-known Crank-Nicolson method, very efficient for standard equation case, is adopted and implemented for the convection-diffusion equation, considering a simple case with constant coefficients. The corresponding numerical results are presented.**

***Keywords*— : convection-difusion equation; difference method; matlab implementation; numerical results**

I. INTRODUCTION

We formulate the convection-difusion equation in its linear form (its simplest form).Find the function u(x,y, t) that satisfies the differential equation

$$\frac{\partial u(x,y,t)}{\partial t} = \alpha \frac{\partial^2 u(x,y,t)}{\partial x^2} + \alpha \frac{\partial^2 u(x,y,t)}{\partial y^2} - \beta \frac{\partial u(x,y,t)}{\partial x} - \beta \frac{\partial u(x,y,t)}{\partial y}$$

$$D = [0,l]x[0,l] \quad , \qquad t > 0$$

with initial condition and boundary conditions respectively

$$u|_D = 0 \quad , t > 0$$
$$u(x,y,0) = f(x,y), \qquad 0 < x < l\,, 0 < y < l$$

The convection-diffusion equation well be simplified with the notation

$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u - \beta \nabla u$$

Where α is the diffusion coefficient and β is colled convection coefficient.

The convection-diffusion equation, also known as the advection-diffusion equation has been used to describe many different physical processes. The convection-diffusion equation is employed as a model for heat transfer and the dynamics of fluids and gasses.

The derivation of the convection-diffusion equation may rely on the principle of superposition, whereby convection and diffusion processes can be treated simultaneously if they are independent. In this study, we focus specifically on the case where the convection and diffusion processes are assumed to be independent. If this assumption is not made, the resulting problem is significantly more complex to analyze.

II. ANALYTICAL SOLUTION OF THE CONVECTION-DIFFUSION EQUATION

In this paper, we have considered the linear convection-diffusion equation with constant coefficients, as given below

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\partial^2 u}{\partial y^2} - \beta \frac{\partial u}{\partial x} - \beta \frac{\partial u}{\partial y}$$

$$D = [0,1]x[0,1] \quad , \quad t > 0 \qquad (1)$$

with initial condition and boundary conditions respectively,

$$u|_D = 0 \quad , t > 0$$
$$u(x,y,0) = f(x,y)\,, \quad 0 < x < 1\,, 0 < y < 1 \qquad (2)$$

The solution is sought in the form

$$u(x,y,t) = X(x)Y(y)T(t) \qquad (\not\equiv 0) \qquad (3)$$

We substitute the solution into equation (1)

$$T'XY = (\alpha X''\text{-}\beta X')YT + X(\alpha Y'' - \beta Y')T$$

Divide both sides by XYT

$$\frac{T'}{T} = \frac{(\alpha X"\text{-}\beta X')}{X} + \frac{(\alpha Y" - \beta Y')}{Y} = -\lambda$$

From the above equation, we derive three equations.

The first equation is for T(t)

$$T\colon \frac{T'}{T} = -\lambda \qquad or \qquad T' + \lambda T = 0 \tag{4}$$

The solution of this first-order differential equation is

$$T(t) = e^{-\lambda t}$$

The other two equations obtained for X(x) and Y(y) have the same solution with different constants, so it is sufficient to solve only one of them. The equations for the variable x and the variable y are

$$\frac{(\alpha X"\text{-}\beta X')}{X} + \frac{(\alpha Y" - \beta Y')}{Y} = -\lambda$$

$$\frac{(\alpha Y" - \beta Y')}{Y} + \lambda = -\frac{(\alpha X"\text{-}\beta X')}{X} = \mu$$

$$X\colon \alpha X" - \beta X' = \mu X$$

$$X\colon \alpha X" - \beta X'\text{-}\mu X = 0 \qquad X(0)=X(1)=0 \tag{5}$$

$$Y\colon (\alpha Y" - \beta Y') + \lambda Y = \mu Y$$

$$Y\colon (\alpha Y" - \beta Y') + (\lambda - \mu) Y = 0 \;\; Y(0) = Y(1) = 0 \tag{6}$$

Let us now solve (5)

$$X\colon \alpha X" - \beta X'\text{-}\mu X = 0 \qquad X(0)=X(1)=0$$

Its characteristic equation is

$$\alpha k^2 - \beta k - \mu = 0 \qquad D = \beta^2 + 4\alpha\mu$$

The roots of this equation are

$k_1 = \frac{\beta - \sqrt{D}}{2\alpha}$ and $k_2 = \frac{\beta + \sqrt{D}}{2\alpha}$

Solving the characteristic equation yields three possible forms for the solution of X(x)

$$X(x) = \begin{cases} C_1 e^{k_1 x} + C_2 e^{k_2 x} & for \quad D > 0 \\ (Ex + F) e^{rx} & for \quad D = 0 \\ e^{\frac{\beta}{2\alpha}x}\left[A sin\left(\frac{-\sqrt{D}}{2\alpha}x\right) + B cos\left(\frac{-\sqrt{D}}{2\alpha}x\right)\right] & for \;\; D < 0 \end{cases}$$

The only equation that satisfies the Dirichlet boundary conditions is

$$X(x) = e^{\frac{\beta}{2\alpha}x}\left[A sin\left(\frac{-\sqrt{D}}{2\alpha}x\right) + B cos\left(\frac{-\sqrt{D}}{2\alpha}x\right)\right] \tag{7}$$

Where D<0 and the roots are imaginary.
We apply the initial condition in (7)

$$X_m(x) = e^{\frac{\beta}{2\alpha}x} sin(mx) \qquad m = 1,2,3,\ldots$$

$$U_m = -\alpha m^2 \pi \pi^2 - \frac{\beta^2}{4\alpha}$$

$$X(x) = \sum_{m=1}^{\infty} A_m \, e^{\frac{\beta}{2\alpha}x} sin(m\pi x)$$

$$A_m = 2\int_0^1 f(x) e^{-\frac{\beta}{2\alpha}x} sin(m\pi x)\, dx$$

Similarly, the solution can be written for *Y(y)*

$$Y(y) = \sum_{n=1}^{\infty} A_n \, e^{\frac{\beta}{2\alpha}y} sin(n\pi y)$$

and

$$\mathrm{X(x)Y(y)} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \mathrm{A_{mn}}\, e^{\frac{\beta}{2\alpha}x} e^{\frac{\beta}{2\alpha}y} \sin(m\pi x)\sin(n\pi y)$$

$$A_{mn} = 4\int_0^1\int_0^1 f(x,y) e^{-\frac{\beta}{2\alpha}x} e^{-\frac{\beta}{2\alpha}y} sin(m\pi x)\, sin(n\pi y)\, dxdy$$

The solution is

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\, e^{\left(\frac{\beta}{2\alpha}x + \frac{\beta}{2\alpha}y + \frac{\beta^2}{2\alpha}\right)t} e^{\frac{\beta}{2\alpha}(x+y)} sin(m\pi x)\, sin(n\pi y)$$

And A_{mn} is

$$A_{mn} = 4\int_0^1\int_0^1 f(x,y) e^{-\frac{\beta}{2\alpha}x} e^{-\frac{\beta}{2\alpha}y} sin(m\pi x)\, sin(n\pi y)\, dxdy$$

III. AN ANALYTICAL EXCEMPLE

In the following, we have presented the results of the analytical solution of the above equation with simple initial and boundary conditions. The results are presented graphically using Matlab.

$$\frac{\partial u}{\partial t} = 0.5\frac{\partial^2 u}{\partial x^2} + 0.5\frac{\partial^2 u}{\partial y^2} - 5\frac{\partial u}{\partial x} - 5\frac{\partial u}{\partial y}$$

$$\mathrm{u(x,y,0) = f(x,y) = \sin(\pi x)\sin(\pi y)}$$

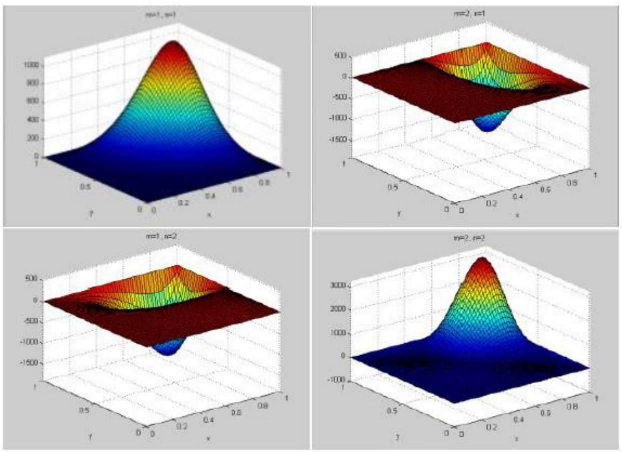

Fig. 1. *The eigenfunctions for: m=1,n=1; m=2,n=1;m=1,n=2;m=2,n=2.*

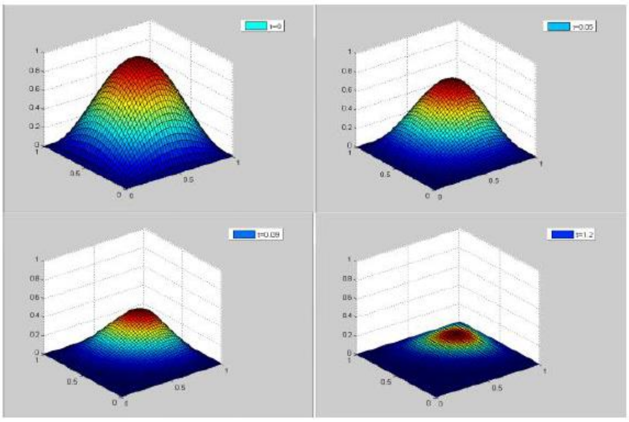

Fig. 2. *The analytical solution at: t=0, t=0.05, t=0.09, t=0.12.*

IV. A Crank-Nicolson Type Method for the Solution of Linear Convection-Diffusion Equation

In this part we have adopted and implemented the Crank-Nicolson method for the solution of linear convection-diffusion equation in two dinmnsions. In several of our earlier works, cited in the references, we have addressed finite difference methods, with particular emphasis on the Crank–Nicolson scheme.

Consider the problem below (1) - (2)

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\partial^2 u}{\partial y^2} - \beta \frac{\partial u}{\partial x} - \beta \frac{\partial u}{\partial y}$$

$$D = [0,1]x[0,1] \quad t > 0$$

with initial condition and boundary conditions respectively

$$u|_D = 0\,, t > 0$$

$$u(x,y,0) = f(x,y)\,, \quad 0 < x < 1\,, 0 < y < 1$$

As a first step, the computational domain is discretized with respect to the independent variables x,y,t.

$$u^j_{m,n} = u(mh, nk, jl) \quad \text{where } h = \Delta x, k = \Delta y \text{ and } l = \Delta t$$

Crank-Nicolson type method can be written

$$\frac{u^{j+1}_{m,n} - u^j_{m,n}}{l} = \frac{\alpha}{2}\left[\left[\frac{u^j_{m+1,n} - 2u^j_{m,n} + u^j_{m-1,n}}{h^2} + \frac{u^{j+1}_{m+1,n} - 2u^{j+1}_{m,n} + u^{j+1}_{m-1,n}}{h^2}\right] + \left[\frac{u^j_{m,n+1} - 2u^j_{m.n} + u^j_{m,n-1}}{k^2} + \frac{u^{j+1}_{m,n+1} - 2u^{j+1}_{m,n} + u^{j+1}_{m,n-1}}{k^2}\right]\right]$$

$$- \frac{\beta}{2}\left[\left[\frac{u^j_{m+1,n} - u^j_{m-1,n}}{2h} + \frac{u^{j+1}_{m+1,n} - u^{j+1}_{m-1,n}}{2h}\right] + \left[\frac{u^j_{m,n+1} - u^j_{m,n-1}}{2k} + \frac{u^{j+1}_{m,n+1} - u^{j+1}_{m,n-1}}{2k}\right]\right]$$

After some algebraic computation we receive

$$\begin{aligned} u^{j+1}_{m,n} &- A_x\left[u^{j+1}_{m+1,n} - 2u^{j+1}_{m,n} + u^{j+1}_{m-1,n}\right] \\ &- A_y\left[u^{j+1}_{m,n+1} - 2u^{j+1}_{m.n} + u^{j+1}_{m,n-1}\right] \\ &+ B_x\left[u^{j+1}_{m+1,n} - u^{j+1}_{m-1,n}\right] \\ &+ B_y\left[u^{j+1}_{m,n+1} - u^{j+1}_{m,n-1}\right] \\ &= u^j_{m,n} + A_x\left[u^j_{m+1,n} - 2u^j_{m,n} + u^j_{m-1,n}\right] \\ &+ A_y\left[u^j_{m,n+1} - 2u^j_{m.n} + u^j_{m,n-1}\right] \\ &- B_x\left[u^j_{m+1,n} - u^j_{m-1,n}\right] \\ &- B_y\left[u^j_{m,n+1} - u^j_{m,n-1}\right] \end{aligned}$$

Where $A_x = \frac{\alpha l}{2h^2}$, $A_y = \frac{\alpha l}{2k^2}$, $B_x = \frac{\beta l}{4h}$ and $B_y = \frac{\beta l}{4k}$

the solution will be stable if the coefficients are all positive.

The last can be written in matrix form as

$$(M_{xy})u^{j+1} = (N_{xy)})u^j$$
$$M_{xy} = (I \otimes M_x + M_y \otimes I)$$

The error is second order in space and time, $O(k^2 + h^2 + l^2)$.

V. A Numerical Example

To illustrate the Crank–Nicolson method, we consider the same example as that used in the analytical solution, subject to identical initial conditions. At the outset, relatively small discretization step sizes are chosen for h, k and l.

Specifically, we take n=45 which yields

$$\mathrm{h} = \mathrm{k} = \frac{1}{45+1} = 0.0217 \text{ and } \mathrm{l} = 0.0001$$

The numerical results are presented graphically in Fig. 3. It is observed that the analytical and numerical solutions yield approximately the same results.

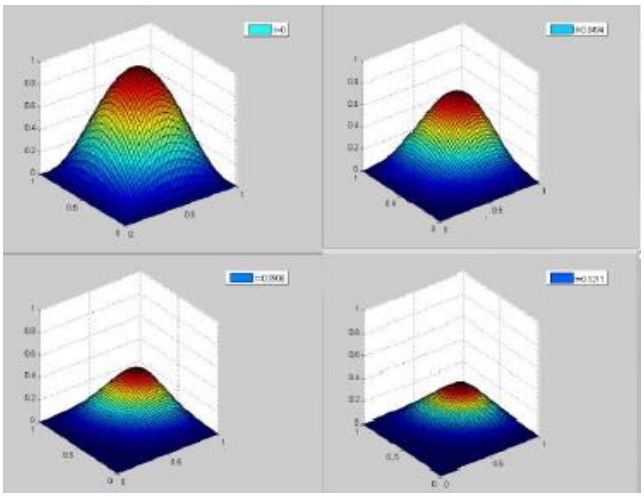

Fig. 3. *The numerical solutions using the Crank-Nicolson method.*

VI. CONCLUSIONS

Many applied mathematics problems lead to solving ordinary differential equations with boundary conditions or even partial differential equations . Since exact solutions of these equations are difficult or almost impossible to obtain, finding approximate methods for these equations provides significant ease. The numerical solution of differential problems involves various difficulties, such as challenges during algebraic procedures or problems related to discretizing the integration domain. Another issue is the stability and convergence of the numerical method used. Finite difference methods have better characteristics and are stable, but generally require more effort and mental resources for their practical implementation. The Crank–Nicolson method is accurate and unconditionally stable for solving linear convection-diffusion equation in two dinmnsions. The analytical and numerical solutions yield approximately the same results.

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