The matching equivalent classes of a graphs

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***Abstract*—Completely characterize the matching equivalent classes of a graphs, by the property of graph's matching polynomial and its maxinum roots.**

Keywords—Matching polynomial; Matching equivalence; Matching uniqueness; The maxinum real roots

I. INTRODUCTION

All graphs considered in this paper are undirected and simple (i.e., loops and multiple edges are not allowed). Let $G=(V(G),E(G))$ be a graph with vertex set $V(G)=\{v_1,v_2,\cdots,v_n\}$ and edge set $E(G)=\{e_1,e_2,\cdots,e_m\}$, where $|V(G)|=n$ is the order and $|E(G)|=m|$ is the size of G. A spanning subgraph H is called a matching of G, if every connected component of H is isolated edge or isolated vertex. k-matching of G is a matching with k edges. In [1] E. J. Farrell denote the matching polynomial as

$$\mu(G,x)=\sum_{k\geq 0}(-1)^k p(G,k)x^{n-2k},$$

where $p(G,k)$ is the number of k-matchings of G.

Two graphs G and H are called matching-equivalent if $\mu(G,x)=\mu(H,x)$, and denoted by $G\sim H$. The disjoint union of two graphs G and H, denoted by $G\bigcup H$, is the graph with vertex set $V(G)\cup V(H)$ and edge set $E(G)\cup E(H)$. kG denotes the disjoint union of k copies of G. Let $M(G)$ be the largest matching root of $\mu(G,x)$. We denote by P_n $(n\geq 1)$, C_n $(n\geq 3)$ a path and a cycle of order n, respectively. A graph $D_{m,n}(m\geq 3,\ n\geq 1)$ is defined as the graph obtained by identifying one end of the path P_{n+1} with a vertex of the cycle C_m. By $T(a,b,c)$ denote the tree which has one 3-degree vertex u and three 1-degree vertices v_1,v_2,v_3 and the distance between u and v_1,v_2,v_3 are a,b,c, respectively. Let P_{n-2} be a path with vertices sequence 1, 2, …, n −2, $I_n(n\geq 6)$ denotes the tree obtained by adding pendant edges at vertices 2 and n – 3 of P_{n-2}, respectively. For a graph G, let G^c be the complement of G. The classes of matching equivalent graphs determined by G under ~ is denoted by [G]. In this paper, we Completely characterize the matching equivalent classes of $I_6\bigcup T(1,1,n)$.

II. PRELIMINARIES

Lemma 2.1 [1] Let G be a graph with k components $G_1,G_2,\ldots,G_k$. Then $\mu(G,x)=\prod_{i=1}^{k}\mu(G_i,x)$.

Lemma 2.2 [1] Let $e=uv\in E(G)$. Then $\mu(G,x)=\mu(G-e,x)-\mu(G-\{u,v\},x)$.

Lemma 2.3 [1] Let G be a connected graph and $u\in V(G)$, $e\in E(G)$. Then $M(G)$ is a single root of $\mu(G,x)$ and $M(G)>M(G-u)$, $M(G)>M(G-e)$.

Lemma 2.4 [3] Let G be a connected graph. Then

(1) $M(G)<2$ if and only if $G\in\Omega_1=\{K_1,P_n,C_n,T(1,1,n),T(1,2,i)(2\leq i\leq 4),D_{3,1}\}$;

(2) $M(G)=2$ if and only if $G\in\Omega_2=\{K_{1,4},T(2,2,2),T(1,3,3),T(1,2,5),I_n,D_{3,2},D_{4,1}\}$.

Lemma 2.5 [4] Let $M(G)<2$. Then graph G is matching uniquely if and only if $G=kK_1\bigcup m_2P_2\bigcup m_3P_3\bigcup[\bigcup_{i\geq 2}m_{2i}P_{2i}]\bigcup[\bigcup_{j\geq 3}n_jC_j]\bigcup dD_{3,1}\bigcup eT(1,2,3)\bigcup fT(1,2,4)$, where $kn_j=m_in_{i+1}=m_2d=m_3d=n_3e=n_{15}e=n_3n_5f=n_5n_9f=0$ and k,m_i,n_j,d,e,f are non-negative integer.

Lemma 2.6 [5] (1) $P_{2m+1}\sim P_m\bigcup C_{m+1}$ $(m\geq 2)$

(2) $T(1,1,n)\sim K_1\bigcup C_{n+2}$

(3) $T(1,2,2)\sim P_2\bigcup D_{3,1}$

(4) $K_1\bigcup C_6\sim P_3\bigcup D_{3,1}$

(5) $K_1\bigcup C_9\sim C_3\bigcup T(1,2,3)$

(6) $K_1\bigcup C_{15}\sim C_3\bigcup C_5\bigcup T(1,2,4)$

Lemma 2.7 [6] (1) $D_{3,2}\sim D_{4,1}$

(2) $K_1\bigcup D_{3,2}\sim I_6$

(3) $T(2,2,2)\sim P_2\bigcup D_{3,2}$

(4) $T(1,3,3)\sim P_3\bigcup D_{3,2}$

(5) $T(1,2,5)\sim P_4\bigcup D_{3,2}$

(6) $K_1\bigcup I_6\sim P_2\bigcup K_{1,4}$

(7) $P_{m-4}\bigcup I_n\sim P_{n-4}\bigcup I_m$ $(m,n\geq 6)$

(8) $I_{2m-3} \square I_m \cup C_{m-3}\ (m \geq 6)$

Lemma 2.8 [5] $G \sim H$ if and only if $G^c \sim H^c$.

III. Main Results

Theorem 3.1 Let $[I_6 \cup T(1,1,n)]$ be The classes of matching equivalent graphs of $I_6 \cup T(1,1,n)$. Then

(1) If $n \neq 1,4,7,13$, $[I_6 \cup T(1,1,n)] = \{P_2 \cup K_{1,4} \cup C_{n+2}, I_6 \cup K_1 \cup C_{n+2}, K_1 \cup D_{3,2} \cup T(1,1,n), 2K_1 \cup D_{3,2} \cup C_{n+2}, K_1 \cup D_{4,1} \cup T(1,1,n), 2K_1 \cup D_{4,1} \cup C_{n+2}\}$;

(2) If $n = 1$, $[I_6 \cup T(1,1,1)] = \{I_6 \cup K_1 \cup C_3, P_2 \cup K_{1,4} \cup C_3, K_{1,4} \cup P_5, K_1 \cup I_9\}$;

(3) If $n = 4$, $[I_6 \cup T(1,1,4)] = \{I_6 \cup K_1 \cup C_6, I_6 \cup P_3 \cup D_{3,1}, K_1 \cup T(1,3,3) \cup D_{3,1}, 2K_1 \cup D_{3,2} \cup C_6, K_1 \cup D_{3,2} \cup P_3 \cup D_{3,1}, 2K_1 \cup D_{4,1} \cup C_6, K_1 \cup D_{4,1} \cup P_3 \cup D_{3,1}\}$;

(4) If $n = 7$, $[I_6 \cup T(1,1,7)] = \{I_6 \cup K_1 \cup C_9, I_6 \cup C_3 \cup T(1,2,3), 2K_1 \cup D_{3,2} \cup C_9, K_1 \cup D_{3,2} \cup C_3 \cup T(1,2,3), 2K_1 \cup D_{4,1} \cup C_9, K_1 \cup D_{4,1} \cup C_3 \cup T(1,2,3)\}$;

(5) If $n = 13$, $[I_6 \cup T(1,1,13)] = \{I_6 \cup K_1 \cup C_{15}, I_6 \cup C_3 \cup C_5 \cup T(1,2,4), 2K_1 \cup D_{3,2} \cup C_{15}, K_1 \cup D_{3,2} \cup C_3 \cup C_5 \cup T(1,2,4), 2K_1 \cup D_{4,1} \cup C_{15}, K_1 \cup D_{4,1} \cup C_3 \cup C_5 \cup T(1,2,4)\}$.

Proof. Let $H \sim I_6 \cup T(1,1,n)$, then $M(H) = M(I_6 \cup T(1,1,n))$, By Lemma 2.4, we obtain $M(H) = M(I_6) = 2$. There must be a connected component H_1 in H belonging to the set Ω_2. Let $H = H_1 \cup H_2$. Thus, we distinguish with the following cases according to $M(I_6)$.

Case 1: If $H_1 = K_{1,4}$, then $I_6 \cup T(1,1,n) \square K_{1,4} \cup H_2$. By $P_2 \cup I_6 \cup T(1,1,n) \square P_2 \cup K_{1,4} \cup H_2 \square K_1 \cup I_6 \cup H_2$, we have $K_1 \cup H_2 \square P_2 \cup T(1,1,n) \square P_2 \cup K_1 \cup C_{n+2}$.

Subcase 1.1: If $n \neq 1$, then $H_2 \square P_2 \cup C_{n+2}$;

Subcase 1.2: If $n=1$, by Lemma 2.6, we have $H_2 \square P_2 \cup C_3 \square P_5$.

Case 2: If $H_1 = T(2,2,2)$, then $I_6 \cup T(1,1,n) \square T(2,2,2) \cup H_2$. By $K_1 \cup I_6 \cup T(1,1,n) \square K_1 \cup T(2,2,2) \cup H_2 \square K_1 \cup P_2 \cup D_{3,2} \cup H_2 \square P_2 \cup I_6 \cup H_2$, we have $P_2 \cup H_2 \square K_1 \cup T(1,1,n) \square 2K_1 \cup C_{n+2}$. By Lemma 2.5 and Lemma 2.6, this case does not exist.

Case 3: If $H_1 = T(1,3,3)$, then $I_6 \cup T(1,1,n) \square T(1,3,3) \cup H_2$. By $K_1 \cup I_6 \cup T(1,1,n) \square K_1 \cup T(1,3,3) \cup H_2 \square K_1 \cup P_3 \cup D_{3,2} \cup H_2 \square P_3 \cup I_6 \cup H_2$, we have $P_3 \cup H_2 \square K_1 \cup T(1,1,n) \square 2K_1 \cup C_{n+2}$.

Subcase 3.1: If $n \neq 4$, By Lemma 2.5 and Lemma 2.6, this case does not exist;

Subcase 3.2: If $n=4$, by Lemma 2.6, we have $P_3 \cup H_2 \square K_1 \cup T(1,1,4) \square 2K_1 \cup C_6 \square K_1 \cup P_3 \cup D_{3,1}$. So $H_2 \square K_1 \cup D_{3,1}$.

Case 4: If $H_1 = T(1,2,5)$, then $I_6 \cup T(1,1,n) \square T(1,2,5) \cup H_2$. By $K_1 \cup I_6 \cup T(1,1,n) \square K_1 \cup T(1,2,5) \cup H_2 \square K_1 \cup P_4 \cup D_{3,2} \cup H_2 \square P_4 \cup I_6 \cup H_2$, we have $P_4 \cup H_2 \square K_1 \cup T(1,1,n) \square 2K_1 \cup C_{n+2}$. By Lemma 2.5 and Lemma 2.6, this case does not exist.

Case 5: If $H_1 = I_m$, then $I_6 \cup T(1,1,n) \square I_m \cup H_2$.

Subcase 5.1: If $n \neq 1,4,7,13$, By Lemma 2.6, we have $I_m \cup H_2 \square I_6 \cup T(1,1,n) \square I_6 \cup K_1 \cup C_{n+2}$. So, if $m = 6$, we have $H_2 \square T(1,1,n) \square K_1 \cup C_{n+2}$;

Subcase 5.2: If $n=1$, By Lemma 2.6 and Lemma 2.7, we have $I_m \cup H_2 \square I_6 \cup T(1,1,1) \square I_6 \cup K_1 \cup C_3 \square K_1 \cup I_9$. So, if $m = 6$, we have $H_2 \square T(1,1,1) \square K_1 \cup C_3$; if $m = 9$, $H_2 \square K_1$;

Subcase 5.3: If $n=4$, By Lemma 2.6, we have $I_m \cup H_2 \square I_6 \cup T(1,1,4) \square I_6 \cup K_1 \cup C_6 \square I_6 \cup P_3 \cup D_{3,1}$. So, if $m = 6$, we have $H_2 \square T(1,1,4) \square K_1 \cup C_6 \square P_3 \cup D_{3,1}$;

Subcase 5.4: If $n=7$, By Lemma 2.6, we have $I_m \cup H_2 \square I_6 \cup T(1,1,7) \square I_6 \cup K_1 \cup C_9 \square I_6 \cup C_3 \cup T(1,2,3)$. So, if $m = 6$, we have $H_2 \square T(1,1,7) \square K_1 \cup C_9 \square C_3 \cup T(1,2,3)$;

Subcase 5.5: If $n=13$, By Lemma 2.6, we have $I_m \cup H_2 \square I_6 \cup T(1,1,13) \square I_6 \cup K_1 \cup C_{15} \square I_6 \cup C_3 \cup C_5 \cup T(1,2,4)$. So, if $m = 6$, we have $H_2 \square T(1,1,13) \square K_1 \cup C_{15} \square C_3 \cup C_5 \cup T(1,2,4)$.

Case 6: If $H_1 = D_{3,2}$, then $I_6 \cup T(1,1,n) \square D_{3,2} \cup H_2$. By Lemma 2.7, we have $K_1 \cup I_6 \cup T(1,1,n) \square K_1 \cup D_{3,2} \cup H_2 \square I_6 \cup H_2$.

Subcase 6.1: If $n \neq 4, 7, 13$, By Lemma 2.6, we have $H_2 \sim K_1 \cup T(1,1,n) \sim 2K_1 \cup C_{n+2}$;

Subcase 6.2: If $n = 4$, By Lemma 2.6, we have $H_2 \sim K_1 \cup T(1,1,4) \sim 2K_1 \cup C_6 \sim K_1 \cup P_3 \cup D_{3,1}$;

Subcase 6.3: If $n = 7$, By Lemma 2.6, we have $H_2 \sim K_1 \cup T(1,1,7) \sim 2K_1 \cup C_9 \sim K_1 \cup C_3 \cup T(1,2,3)$;

Subcase 6.4: If $n = 13$, By Lemma 2.6, we have $H_2 \sim K_1 \cup T(1,1,13) \sim 2K_1 \cup C_{15} \sim K_1 \cup C_3 \cup C_5 \cup T(1,2,4)$;

Case 7: If $H_1 = D_{4,1}$. By $D_{3,2} \sim D_{4,1}$, it is similar to Case 6. The proof is omitted.

The proof of Theorem 3.1 is complete.

By Theorem 3.1 and Lemma 2.8, we have Theorem 3.2:

Theorem 3.2 Let $[(I_6 \cup T(1,1,n))^c]$ be The classes of matching equivalent graphs of $(I_6 \cup T(1,1,n))^c$. Then

(1) If $n \neq 1, 4, 7, 13$, $[(I_6 \cup T(1,1,n))^c] = \{(P_2 \cup K_{1,4} \cup C_{n+2})^c, (I_6 \cup K_1 \cup C_{n+2})^c, (K_1 \cup D_{3,2} \cup T(1,1,n))^c, (2K_1 \cup D_{3,2} \cup C_{n+2})^c, (K_1 \cup D_{4,1} \cup T(1,1,n))^c, (2K_1 \cup D_{4,1} \cup C_{n+2})^c\}$;

(2) If $n = 1$, $[(I_6 \cup T(1,1,1))^c] = \{(I_6 \cup K_1 \cup C_3)^c, (P_2 \cup K_{1,4} \cup C_3)^c, (K_{1,4} \cup P_5)^c, (K_1 \cup I_9)^c\}$;

(3) If $n = 4$, $[(I_6 \cup T(1,1,4))^c] = \{(I_6 \cup K_1 \cup C_6)^c, (I_6 \cup P_3 \cup D_{3,1})^c, (K_1 \cup T(1,3,3) \cup D_{3,1})^c, (2K_1 \cup D_{3,2} \cup C_6)^c, (K_1 \cup D_{3,2} \cup P_3 \cup D_{3,1})^c, (2K_1 \cup D_{4,1} \cup C_6)^c, (K_1 \cup D_{4,1} \cup P_3 \cup D_{3,1})^c\}$;

(4) If $n = 7$, $[(I_6 \cup T(1,1,7))^c] = \{(I_6 \cup K_1 \cup C_9)^c, (I_6 \cup C_3 \cup T(1,2,3))^c, (2K_1 \cup D_{3,2} \cup C_9)^c, (K_1 \cup D_{3,2} \cup C_3 \cup T(1,2,3))^c, (2K_1 \cup D_{4,1} \cup C_9)^c, (K_1 \cup D_{4,1} \cup C_3 \cup T(1,2,3))^c\}$;

(5) If $n = 13$, $[(I_6 \cup T(1,1,13))^c] = \{(I_6 \cup K_1 \cup C_{15})^c, (I_6 \cup C_3 \cup C_5 \cup T(1,2,4))^c, (2K_1 \cup D_{3,2} \cup C_{15})^c, (K_1 \cup D_{3,2} \cup C_3 \cup C_5 \cup T(1,2,4))^c, (2K_1 \cup D_{4,1} \cup C_{15})^c, (K_1 \cup D_{4,1} \cup C_3 \cup C_5 \cup T(1,2,4))^c\}$.

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