Examination of QR Decomposition and the Singular Value Decomposition Methods

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***Abstract*— For any problem to be solved, a method that adequately accommodates its complexity range must first be examined and selected. Some methods cannot take care of the order or degree of the problems at hand. Yet the solution accommodation of few methods has been proven theoretical, namely: the QR Decomposition and Singular Value Decomposition (SVD). This paper compares the QR and SVD methods and their speed of convergence. The QR Decomposition works faster and could be enhanced for better accuracy than the SVD but it is limited when a matrix is close to rank-deficient.**

Keywords*— *QR Decomposition, Singular Value Decomposition (SVD)

I. INTRODUCTION

Numerical analysis has found application in solving real problems; particularly problems ranging from least square solutions to image and signal processing, inverse scattering, inverse problems, and so on. Methods propounded and proven have been used in recent times to reduce and solve the complexities associated with practical entities aside the proven theories. Gauss-Jordan, LU (Lower Upper) Decomposition, Cholesky Factorization, Singular Value Decomposition (SVD), QR Decomposition are methods that have found relative use in the real-time application and industry. Yet as vibrant as some methods may be on paper, they have often lacked the capacity and numerical stability to prove systems of difficult problems.

For any problem to be solved, a method that adequately accommodates its complexity range must first be examined and selected. Some methods cannot take care of the order or degree of the problems at hand. Yet the solution accommodation of few methods has been proven theoretical, namely: the QR Decomposition and Singular Value Decomposition (SVD). This paper compares the QR and SVD methods as used in communications engineering problems and their speed of convergence.

II. COMPUTING THE QR

A. *QR DECOMPOSITION*

The QR Decomposition of an *m x n*- dimensional complex-valued matrix *A* is defined as

$A = QR$

where *Q* is *(m x n)* orthonormal column matrix, and *R* is *(n x n)* upper triangular matrix. The QR decomposition avoids the shortcomings of Normal Equations Method. The Normal Equations Method (e.g. Cholesky Factorization) has poor system conditioning and instability [1] but seems to be superior to classical Gram-Schmidt orthonormalization process. The QR decomposition uses the orthogonal methods, where $Q \in R^{mxn}$ matrix is the orthonormal columns of matrix *A*.

B. *ORTHOGONAL MATRIX*

A matrix is orthogonal if its columns are unit length and mutually perpendicular [2]. A matrix $Q \in R^{mxn}$ with $m \geq n$, has orthonormal columns if all columns in *Q* are orthogonal to every other column and are normalized [1]. If a matrix *Q* is orthogonal, then

$Q^TQ = QQ^T = I$

where Q^T is the transpose of matrix *Q*, and *I* is the Identity matrix. Then it also holds that $Q^{-1} = Q^T$, therefore, if *Q* is orthogonal, then Q^T is also orthogonal.

III. COMPUTATION OF QR DECOMPOSITION

QR decomposition is done by reducing *A* to an upper triangular matrix *Q* by applying the orthogonal transformations: Gram-Schmidt or Householder Reflectors. For the purpose of this work, the Householder reflection is closely examined.

A. *HOUSEHOLDER TRANSFORMATION*

A Householder matrix—sometimes called Householder reflection [3]—is defined by a nonzero vector v, and it is just a reflection along the v direction [2]. It has the form $H \in R^{nxn}$, where $H = I - 2\frac{vv^T}{v^Tv}$

Given a vector x, the reflection that can transform x into a direction parallel to some unit vector y can be found to be:

If $H = I - 2\frac{vv^T}{v^Tv}$, then we can have $Hx = I - 2\frac{vv^T}{v^Tv}x$

B. Applying Householder Transformation

The QR transformation is first to zero out everything in the first column below the (1, 1) entry of a matrix A. Given a (4x3) matrix A as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$$

A reflection H_1 is constructed such that

$$H_1A = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} \\ 0 & a_{42}^{(1)} & a_{43}^{(1)} \end{bmatrix},$$

and next a reflection is used to nullify $a_{i2}^{(1)}$, $i = 3{:}\ 4$

$P_2 \begin{bmatrix} a_{22}^{(1)} \\ a_{32}^{(1)} \\ a_{42}^{(1)} \end{bmatrix} = \begin{bmatrix} r_{22} \\ 0 \\ 0 \end{bmatrix}$, to move on, apply the transformation

$H_2 = \begin{bmatrix} 1 & 0 \\ 0 & P_2 \end{bmatrix}$ to obtain

$$H_2H_1A = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & a_{33}^{(2)} \\ 0 & 0 & a_{43}^{(2)} \end{bmatrix}.$$

This is repeated on the third column to finally have

$$H_3H_2H_1A = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \\ 0 & 0 & 0 \end{bmatrix}.$$

Q is obtained from the Householder Reflection as,
$Q = (H_1H_2 \ldots H_n)^{-1} = (H_1H_2 \ldots H_n)^T = H_1^TH_2^T \ldots H_n^T = H_1H_2 \ldots H_n$.
It must be noted that Q does not need explicit construction. It is embedded in the Householder reflection. It makes the method easier to use.

IV. SINGULAR VALUE DECOMPOSITION (SVD)

SVD is a factorization of a matrix A into the product of three matrices $A = UDV^T$. Often, the nonzero singular values of A are computed by an eigenvalue computation for the normal matrix A^TA. However, this method has been found to be numerically unstable as used in the Normal Equations (e.g. Cholesky). Optionally, the Householder reflection is used to transform A to a bi-diagonal form and then into a diagonal form using two sequences of orthogonal matrices. Take note that has the latter method is done, computing complexity of SVD has been found to be approximately $2mn^2 + 11n^3$ [3].

V. COMPARISON OF THE QR AND SVD USING A RECTANGULAR MATRIX

A. SVD SOLUTION

For the purpose of comparison, an example of a rectangular matrix is given and conclusions are drawn from the arrays of solution steps. Considering a given matrix [13]

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

To compute the SVD, matrix A must be multiplied by its transpose to obtain a square matrix. Since the eigenvector method is the elementary means of calculating SVD aside the householder's transformation, eigenvalue of a non-square matrix cannot be found, therefore A is multiplied by its transpose. The result is

$$\begin{bmatrix} 18 & -11 & 5 & 2 \\ -11 & 21 & 13 & -3 \\ 5 & 13 & 25 & -3 \\ 2 & -3 & -3 & 2 \end{bmatrix}.$$

The matrix's diagonal acts as a mirror with the lower triangle equal to the upper triangle. The eigenvalue of the matrix was found to be $\begin{bmatrix} 37.6 \\ 25.5 \\ 2.1 \\ 0.8 \end{bmatrix}$. These values are the singular vectors, which are used as the diagonals of D, also it should be noted that the eigenvalues of AA^T and A^TA will always be the same. Next, the singular vector values are used to obtain the eigenvectors, which is $\begin{bmatrix} 0.2 & -0.8 & -0.4 & 0.4 \\ -0.7 & 0.3 & -0.4 & 0.6 \\ -0.7 & -0.6 & 0.4 & -0.3 \\ 0.1 & 0.0 & 0.7 & 0.7 \end{bmatrix}$.

Having obtained this, the Gram-Schmidt orthonormalization method is used to normalize the column vectors to give the value of U as

$$\begin{bmatrix} -0.2 & -0.8 & 0.4 & 0.4 \\ 0.7 & 0.3 & 0.6 & -0.4 \\ 0.7 & -0.6 & -0.3 & 0.4 \\ -0.1 & 0.0 & 0.7 & 0.7 \end{bmatrix}.$$

The value of V is obtained from A^TA, which equals $\begin{bmatrix} 4 & 6 & 0 \\ 6 & 34 & -4 \\ 0 & -4 & 24 \end{bmatrix}$. The Eigen vector obtained is

$$\begin{bmatrix} -1.0 & -0.2 & 0.1 \\ 0.2 & -0.9 & 0.3 \\ 0.0 & 0.3 & 1.0 \end{bmatrix}$$

Performing the Gram-Schmidt Orthonormalization method on the eigenvector matrix,

$$V = \begin{bmatrix} -0.98 & -0.196 & 0.065 \\ 0.196 & -0.98 & 0.522 \\ 0.0 & 0.0 & 0.568 \end{bmatrix}$$

and its transpose

$$V^T = \begin{bmatrix} -0.98 & 0.196 & 0.0 \\ -0.196 & -0.98 & 0.0 \\ 0.065 & 0.522 & 0.568 \end{bmatrix}$$

and using the SVD formula we have

$$A = UDV^T =$$

$$\begin{bmatrix} -0.2 & -0.8 & 0.4 & 0.4 \\ 0.7 & 0.3 & 0.6 & -0.4 \\ 0.7 & -0.6 & -0.3 & 0.4 \\ -0.1 & 0.0 & 0.7 & 0.7 \end{bmatrix} \begin{bmatrix} 37.6 & 0 & 0 & 0 \\ 0 & 25.5 & 0 & 0 \\ 0 & 0 & 2.1 & 0 \\ 0 & 0 & 0 & 0.8 \end{bmatrix} \begin{bmatrix} -0.98 & 0.196 & 0.0 \\ -0.196 & -0.98 & 0.0 \\ 0.065 & 0.522 & 0.568 \end{bmatrix}$$

where $D = \begin{bmatrix} 37.6 & 0 & 0 & 0 \\ 0 & 25.5 & 0 & 0 \\ 0 & 0 & 2.1 & 0 \\ 0 & 0 & 0 & 0.8 \end{bmatrix}$

VI. QR SOLUTION

The reflector vector is calculated as $v_1 = a_1 - \text{sign}(a_{11})\|a_1\|e_1$, which is $\begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$. Next, the Householder matrix is calculated as $H_1 = I - 2\frac{v_1 v_1^T}{v_1^T v_1} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}$. H_1 is thereafter multiplied by A as $H_1A = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & -5 & 2 \end{bmatrix}$.

Then $v_2 = \begin{pmatrix} 0 \\ 0 \\ -5 \end{pmatrix} - \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ -5 \end{pmatrix}$, and $H_2 = I - 2\frac{V_2 V_2^T}{V_2^T V_2} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

Therefore,

$$H_2H_1A = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

The R factor is $\begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{bmatrix}$,

and the

$$Q = \begin{bmatrix} 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}$$

i.e. $Q = H_2H_1$ (as stated earlier).

VII. OBSERVATION OF THE METHODS

(1) The Singular Value Decomposition method (SVD) can be difficult to use on large matrices. For the example given above, there were observed discrepancies when the SVD was solved manually without a mathematical tool. When the matrix calculator ran the SVD for the given matrix, these values were obtained:

$$U = \begin{bmatrix} -0.3 & 0.8 & -0.2 \\ 0.7 & -0.1 & -0.4 \\ 0.6 & 0.6 & 0.2 \\ -0.1 & 0.0 & -0.9 \end{bmatrix},$$

$$D = \begin{bmatrix} 6.0 & 0.0 & 0.0 \\ 0.0 & 4.9 & 0.0 \\ 0.0 & 0.0 & 1.4 \end{bmatrix},$$

$$V^T = \begin{bmatrix} 0.1 & 1.0 & -0.3 \\ 0.3 & 0.2 & 0.9 \\ -1.0 & 0.2 & 0.2 \end{bmatrix}$$

This deviation is wide; this can be explained because of the large matrix involved and the calculation being done by hand.

(2) The SVD traditional method solves for the eigenvalue, λ. This is obtained from the characteristic polynomial. Solution becomes cumbersome for large matrices. Another issue to note is that, the eigenvalue of a matrix cannot be calculated without the matrix being in a symmetrical form. The QR decomposition isolates the column of the matrix at the beginning of computation. An issue to note about eigenvalues is that when complexity is involved, any algorithm to be used must be iterative, if not stability will not be guaranteed. Taking note that stability (numerical) is a major issue that any employed method must address; if the algorithm cannot achieve this, it defeats the purpose of usage.

An observed characteristic of the QR decomposition is this: when the component (i.e. property) of a matrix is examined closely, individual attributes or characteristics will be highlighted and solution becomes easier. The QR works on matrix components unlike the SVD, which does not exploit the properties of matrix A but only that of A^TA and AA^T (using the eigenvector approach).

(3) The Householder reflector, when performed on matrix A creates a reflection of vector x. The Householder matrices obtained are multiplied together to give the matrix Q. Simply put, Q is obtained directly from the Householder matrices, and hence, faster computation is done. In SVD, when the Householder reflection is employed, it transforms A to a bi-diagonal form and then into diagonal form using two sequence orthogonal matrices. This is an extension of the QR factorization. The extension of householder transformations to bi-diagonalization is called Golub-Kahan bi-diagonalization [4]. The Householder reflection is applied to the right and left sides of A

$$A \rightarrow U^*_n \ldots U^*_1 A V_1 \ldots V_{n-2}.$$

Reflectors applied to the left introduce column zeros beneath the diagonal while those applied to the right introduce row zeros. At the end of the algorithm, n reflectors have been applied to the left and n-2 reflectors had been applied to the right [5], leaving A to be bi-diagonal. Since Householder transformation on two sides can be tricky, therefore the Lanczos recurrence is an alternative method for bi-diagonalization [6]. The Lanczos recurrence is not within the scope of this work.

The next phase is to compute the SVD of the bi-diagonal matrix. This phase is not an analytic algorithm [6]. There are methods available that work well for both well-conditioned and ill-conditioned matrices but at high computational cost [7]. It should also be noted that there are divide and conquer algorithms employed in diagonalization that are relatively efficient and robust [8].

Considering the argument stated earlier, it could be observed that the SVD is expensive and cumbersome. The computational time employed to perform the bi-diagonalization and diagonalization counts in usage of the method. Either the Eigen approach or Bi-diagonal approach is employed; the use of the SVD must be justified. The QR Decomposition works faster and could be enhanced for better accuracy than the SVD [9].

VIII. ENGINEERING APPLICATIONS OF SVD AND QR

Despite the observations made, SVD and QR decomposition have found useable application in Engineering.
SVD has found wide application in image compression and processing [10]; it has been found to be useful in image forensic. Aside this, [11] has adapted SVD to develop a new sub-band decomposition and multi-resolution representation of digital colour images.
In Control Engineering, it has been used in the examination of dynamic behavior and optimization of systems.[12]
In receiver design of Multiple Input Multiple Output (MIMO) systems, SVD has been used to obtain the channel capacity estimation and computation of the pseudoinverse of the channel matrix [13][14].
SVD has been used as a dimension reduction technique. SVD forms a computational base for Principal Component Analysis [15] and as a determinant of low-dimension approximation to highly dimensional data. Also, in [16], SVD was used as a dimensionality reduction technique for an improved term frequency -inverse document frequency (TF-IDF) algorithm in feature extraction on Twitter.

In the area of detection, location and characterization of damage in structural and mechanical systems, SVD has widely been employed as an engineering tool for effective and efficient damage control method [17]. In [18], it was combined with time domain features and fuzzy logic system. SVD was used in the reduction of the feature matrix and selection of most stable vectors.
In fault diagnosis of rolling bearings, SVD has been adaptively modified for fault feature detection [19]. Furthermore, in [20] a fault diagnosis approach was developed using Wavelet Packet Transform, Support Vector Machine and SVD. In this approach, SVD was used to obtain the singular value vectors as feature vectors which are classified by the SVM.

In Hardware security, SVD has found usage in the detection of ghost circuitry and gate characterization [21].

In reverse engineering, SVD has been applied in gene networks. It was used to reduce a large and sparse network, by constructing a family of candidate solutions [22].
In adaptive audio watermarking, singular values in the SVD of wavelet domain blocks were applied with quantization-index-modulation foe embedding a watermark [23].

In cutting process, SVD was leveraged as a tool to obtain the vector of coefficients of a quadratic sub-expression embedded in a Group Method of Data Handling (GMDH)-type network. The GMDH-type neural networks are employed for cutting process of plates by shaped charge [24].
Higher order Singular Value Decomposition has been in real time performance improvement of engineering control units for ST engine [25].
SVD has further found application in the study of account emission data. A method of acoustic signals identification based on singular spectral was presented in [26]
Due to the computational expense of SVD, its application in robotics has been limited in time past to analysis of kinematic and dynamic properties of robotic manipulators. Yet, SVD has been recently applied to real-time problem solving in robotics [27].
The mathematical strength of SVD has also been harnessed with Genetic Algorithm in predicting the discharge coefficient in a side weir. SVD helps in the linear parameter computation of the adaptive neuro-fuzzy inference system [28].
In compressing large datasets of streamflow. SVD was found in [29] to be an efficient forecasting tool over principal component Analysis.

In the area of Cloud computing, SVD has been used to develop an analytic algorithm that helps user application to determine the best service provider. SVD was used as a ranking and mapping technique in cloud computing [30].

In seismic data analysis, SVD has also found local usage. In [31], SVD was used to improve the signal-to-noise ratio of stacked and unstacked seismic data.
In Artificial Intelligence, SVD has been used has a method for selection, classification, clustering, and modeling of DNA microarray data [32].

As SVD has been singularly explored in most areas of engineering science and applications, QR decomposition has also found appreciable and applicable utilization in Engineering
QR decomposition has been applied in Principal Component Analysis. It was found in [33] that QR method was numerically stable and computationally efficient than SVD.
QR decomposition has also be useful in the reduction of set of differential and algebraic equations to state space form for mechanical system [34]
In MIMO systems, QR Decomposition is a main MIMO technique that is used for decomposing matrices into a product of orthonormal matrices and triangular matrices. several modular methods have been developed for its swift implementation [35][13].
In linear discriminant Analysis (LDA), QR decomposition has been examined and proposed as a fast algorithm for singular scatter matrices [36]. In the study of systems, QR decomposition was applied to form a new identification method for nonlinear time-varying multiple- degree-of-freedom (MDOF) dynamic

system for location and estimation of nonlinearities [37].

IX. CONCLUSION

The QR Decomposition works faster and it is a method considered to be straightforward to use and deal with. Though having the advantage of being fast, what limits the QR decomposition is when the matrix under study is close to rank-deficient [38]. Yet, it stands that the SVD is a brilliant tool for inconsistent, underdetermined linear systems and least square problems.

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