The Construction Of Extended Second Derivative Block Backward Differentiation Formula For Numerical Solutions Of First Order Delay Differential Equations

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***Abstract*— A suitable extended second derivative backward differentiation formulae method in block form was constructed and applied in determining the numerical solutions of some first order delay differential equations (DDEs) without the introduction of interpolation techniques in evaluating the delay term. The continuous expressions of each step number of this method were investigated to obtain the discrete schemes through the linear multistep collocation technique by matrix inversion procedure. The obtained discrete schemes were tested which clearly revealed the order and error constants, consistency, zero stability, convergence and region of absolute stability of this method. After the application of this method in solving some first order delay differential equations, the results obtained proved that the higher step number $k=4$ integrated with extended future point performed better than the lower step numbers $k=3$ and 2 integrated with extended future points when compared with the exact solutions and other existing methods.**

Mathematical Subject Classification: 34K28; 65F30

Keywords— First order delay differential equations, block method, extended future points, backward differentiation formulae

I. INTRODUCTION

Renowned scholars have shown the applications of delay differential equations in many real life situations involving celestial and quantum mechanics, nuclear and theoretical physics, astrophysics, quantum chemistry, molecular dynamics, engineering, medicine and economic dynamics and control. The evolution of delay differential equations (DDEs) involves the knowledge of both the past values and current values of the quantities being modeled.

In this paper, we shall focus on obtaining the numerical solutions of the first order delay differential equations (DDEs) developed by Bellman et al (1963);

$$x'(t) = f(t, x(t), x(t-\tau)), \quad \text{for } t > t_0, \tau > 0$$
$$x(t) = \alpha(t), \text{ for } t \le t_0 \qquad (1)$$

where $\alpha(t)$ is the initial function, τ is called the delay, $(t-\tau)$ is called the delay argument and $x(t-\tau)$ is the solution of the delay term using extended second derivative block backward differentiation formulae methods. Most real life phenomena involve delay and have been studied by many researchers numerically and analytically which revealed the numerous advantages of DDEs over ODEs. Such research activities can be found in the works of these notable scholars; [1] applied interpolation method to investigate the delay argument while solving delay differential equations in a predictor-corrector mode. [2] adopted linear multistep methods for the numerical solution of initial value problem for stiff delay differential equations with several constant delays with Nordsieck's interpolation technique. [3] delay differential equations was used in the modeling of El Nino temperature oscillations in the Equatorial Pacific to determine the model single-species population growth. In electrical circuits, delays are introduced because it takes time for a signal to travel through a transmission line. [4] formulated the algorithms for solving non-stiff DDEs with time dependent delays using predictor-corrector version of the one-step collocation method at Gaussion points. [5], constructed a numerical method for linear and non-linear higher order delay differential equations through Adomain decomposition method. [6], applied the direct two-point fourth and fifth order multistep block method in the form of Adams- Moulton Method in predictor-corrected (PECE) mode to compute the numerical solutions at two points simultaneously for second order delay differential equations. [7], used linear multistep approach in solving delay differential equations with the aid of interpolation techniques in estimating the delay argument. [8] applied implicit 2-point Block Backward Differential Formulae to solve a set of delay differential equations with the help of interpolation technique in approximating the delay term. [9] used the Zhou method to obtain approximate-analytical solutions of certain system of functional differential equations (SFDEs) induced by proportional delays.[10] procured the results of some system of time-fractional differential equations (TFDDEs) using proportional delays. [11] examined the results of fractional order delay differential equations by applying Shifted Legendre polynomials with Collocation and Galerkin settings. [12] investigated the numerical solution of stiff differential equations using a class of fifth order block hybrid Adams Moulton's Method to determine the accuracy and efficiency of each step number incorporated with off-grid points. [13] implemented block simpson's methods in solving first order delay differential equations without using any interpolation techniques in examining the delay term.[14] implemented third derivative block backward differentiation formulae for numerical solutions of first order delay differential equations without interpolation techniques in investigating the delay argument. [15] solved first order delay differential equations using multiple off-grid hybrid block simpson's methods without the application of any interpolation technique in estimating the delay term.[16] solved first DDEs using extended block backward differentiation formulae for efficiency of the numerical solution without the introduction of interpolation techniques in finding the delay argument.

One of the limitations in applying these interpolation techniques by [17] is that the numerical methods to be implemented in solving DDEs should be the same with the interpolating polynomials which is very hard to carry out; otherwise, the accuracy of the method will not be preserved. It is crucial that in the estimation of the delay term, introducing a reliable and coherent formula shall be highly accepted.

To overcome this limitation caused by applying interpolation techniques in estimating the delay term; we implement the idea of the sequence formed by [18] which we merge into the first order delay differential equations. Then we applied extended second derivative block backward differentiation formulae methods to solve some first order delay differential equations containing the estimated delay term to enhance the performance of the existing method on second derivative BBDF been studied by [19] in terms of efficiency, accuracy, consistency, convergence and region of absolute stability at constant step width b .

II. CONSTRUCTION PROCEDURE

A. *Construction of Second Derivative Backward Differentiation Formulae Method*

In [20], the k-step generalized Backward Differentiation Formulae Methods was constructed as

$$\sum_{e=0}^{s}\alpha_e(x)y_{d+e}=b\beta_e(x)f(x_e,y(x_e)) \qquad (2)$$

And its second derivative is expressed as;

$$\sum_{e=0}^{s}\alpha_e(x)y_{d+e}=b\beta_e(x)f(x_e,y(x_e)+b^2\gamma_e(x)g(x_e,y(x_e) \qquad (3)$$

And its extended second derivative is expressed as;

$$\sum_{e=0}^{s}\alpha_e(x)y_{d+e}=b\beta_e(x)f(x_e,y(x_e)+b^2\gamma_e(x)g(x_e,y(x_e)+b^2\delta_e(x)q(x_e,y(x_e) \qquad (4)$$

where $\alpha_e(x)$, $\beta_e(x)$, $\gamma_e(x)$ and $\delta_e(x)$ are continuous coefficients of the method defined as

$$\alpha_e(x)=\sum_{u=0}^{g+l-1}\alpha_{e,u+1}x^u \text{ for } e=\{0,1,...,g-1\} \qquad (5)$$

$$b\beta_e(x)=\sum_{u=0}^{g+l-1}b\beta_{e,u+1}x^u \text{ for } e=\{0,1,...,l-1\} \qquad (6)$$

$$b^2\gamma_e(x)=\sum_{u=1}^{s}b^2\gamma_{e,u+1}x^u \text{ for } e=\{0,1,...,l-1\} \qquad (7)$$

$$b^2\delta_e(x)=\sum_{u=1}^{s}b^2\gamma_{e,u+1}x^u \text{ for } e=\{0,1,...,l-1\} \qquad (8)$$

where $x_0, \ldots, x_{l-1}$ are the l collocation points, x_{d+e} , $e=0,1,2,...,g-1$ are the g arbitrarily chosen interpolation points and b is the fixed step size and $s=k$ is the step number.

To get continuous coefficients $\alpha_e(x)$, $\beta_e(x)$, $\gamma_e(x)$ and $\delta_e(x)$ of the propose method, we developed a matrix equation of the form

$$ZJ=\mathrm{I} \qquad (9)$$

where I represents the elementary matrix of dimension $(g+l)\times(g+l)$ and Z and J are matrices presented as

$$Z=\begin{pmatrix} \alpha_{0,1} & \alpha_{1,1} & \cdot & \alpha_{g-1,1} & b\beta_{0,1} & \cdot & b\beta_{l-1,1} & b^2\gamma_{0,1} & \cdot & b^2\gamma_{l-1,1} & b^2\delta_{0,1} & \cdot & b^2\delta_{l-1,1} \\ \alpha_{0,2} & \alpha_{1,2} & \cdot & \alpha_{g-1,2} & b\beta_{0,2} & \cdot & b\beta_{l-1,2} & b^2\gamma_{0,2} & \cdot & b^2\gamma_{l-1,2} & b^2\delta_{0,2} & \cdot & b^2\delta_{l-1,2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{0,e+l} & \alpha_{1,e+l} & \cdot & \alpha_{g-1,e+l} & b\beta_{0,e+l} & \cdot & b\beta_{l-1,e+l} & b^2\gamma_{0,e+l} & \cdot & b^2\gamma_{l-1,e+l} & b^2\delta_{0,e+l} & \cdot & b^2\delta_{l-1,e+l} \end{pmatrix} \qquad (10)$$

$$J=\begin{pmatrix} 1 & x_d & x_d^2 & x_d^3 & x_d^4 & \cdot\,\cdot & x_d^{g+l-1} \\ 1 & x_{d+1} & x_{d+1}^2 & x_{d+1}^3 & x_{d+1}^4 & \cdot\,\cdot & x_{d+1}^{g+l-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\,\cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\,\cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\,\cdot & \cdot \\ 1 & x_{d+g-1} & x_{d+g-1}^2 & x_{d+g-1}^3 & x_{d+g-1}^4 & \cdot\,\cdot & x_{d+g-1}^{g+l-1} \\ 0 & 1 & 2x_d & 3x_d^2 & 4x_d^3 & \cdot\,\cdot & (g+l-1)x_d^{g+l-2} \\ 0 & 1 & 2x_{d+1} & 3x_{d+1}^2 & 4x_{d+1}^3 & \cdot\,\cdot & (g+l-1)x_{d+1}^{g+l-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\,\cdot & \cdot \\ 0 & 1 & 2x_{d+g-1} & 3x_{d+g-1}^2 & 4x_{d+g-1}^3 & \cdot\,\cdot & (g+l-1)x_{d+g-1}^{g+l-2} \\ 0 & 0 & 2 & 6x_d & 12x_d^2 & \cdot\,\cdot & (g+l-1)(g+l-2)x_d^{g+l-3} \\ 0 & 0 & 2 & 6x_{d+1} & 12x_{d+1}^2 & \cdot\,\cdot & (g+l-1)(g+l-2)x_{d+1}^{g+l-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\,\cdot & \cdot \\ 0 & 0 & 2 & 6x_{d+g-1} & 12x_{d+g-1}^2 & \cdot\,\cdot & (g+l-1)(g+l-2)x_{d+g-1}^{g+l-3} \end{pmatrix} \quad (11)$$

From (9) the columns of $Z = J^{-1}$ give the continuous coefficients of the continuous scheme (4).

B. Construction of Extended Second Derivative Block Backward Differentiation Formulae Method for k = 2

Here, we incorporate one extended future point at $x = x_{d+3}$ as a collocation point, thus the interpolation points, $g = 2$ and the collocation points $l = 3$ are considered,

Therefore, (4) becomes

$$y(x) = \alpha_0(x)y_d + \alpha_1(x)y_{d+1} + b\beta_2(x)f_{d+2} + b^2\gamma_2(x)g_{d+2} + b^2\delta_3(x)q_{d+3} \quad (12)$$

The matrix J in (9) becomes

$$J=\begin{pmatrix} 1 & x_d & x_d^2 & x_d^3 & x_d^4 \\ 1 & x_d+b & (x_d+b)^2 & (x_d+b)^3 & (x_d+b)^4 \\ 0 & 1 & 2x_d+4b & 3(x_d+2b)^2 & 4(x_d+2b)^3 \\ 0 & 0 & 2 & 6x_d+12b & 12(x_d+2b)^2 \\ 0 & 0 & 2 & 6x_d+18b & 12(x_d+3b)^2 \end{pmatrix} \quad (13)$$

The inverse of the matrix $Z = J^{-1}$ is computed using Maple 18 from which the continuous scheme is obtained using (4) and evaluating it at $x = x_{d+2}, x = x_{d+3}$ and its derivative at $x = x_{d+1}$,the following discrete schemes are obtained

$$y_{d+1} = \frac{101}{60}b^2g_{d+2} - \frac{17}{60}b^2q_{d+3} + \frac{29}{10}bf_{d+1} - \frac{19}{10}bf_{d+2} + y_d$$

$$y_{d+2} = -\frac{3}{29}y_d + \frac{32}{29}y_{d+1} + \frac{26}{29}bf_{d+2} - \frac{34}{87}b^2g_{d+2} + \frac{4}{87}b^2q_{d+3}$$

$$y_{d+3} = -\frac{4}{29}y_d + \frac{33}{29}y_{d+1} + \frac{54}{29}bf_{d+2} + \frac{1}{29}b^2g_{d+2} + \frac{5}{29}b^2q_{d+3} \quad (14)$$

C. Construction of Extended Second Derivative Block Backward Differentiation Formulae Method for k = 3

In this case, we incorporate one extended future point at $x = x_{d+4}$ as a collocation point, thus the interpolation points, $g = 3$ and the collocation points $l = 3$ are considered,

Therefore, (4) becomes

$$y(x) = \alpha_0(x)y_d + \alpha_1(x)y_{d+1} + \alpha_2(x)y_{d+2} + b\beta_3(x)f_{d+3} + b^2\gamma_3(x)g_{d+3} + b^2\delta_4(x)q_{d+4} \quad (15)$$

Also the matrix J in (9) becomes

$$J=\begin{pmatrix} 1 & x_d & x_d^2 & x_d^3 & x_d^4 & x_d^5 \\ 1 & x_d+b & (x_d+b)^2 & (x_d+b)^3 & (x_d+b)^4 & (x_d+b)^5 \\ 1 & x_d+2b & (x_d+2b)^2 & (x_d+2b)^3 & (x_d+2b)^4 & (x_d+2b)^5 \\ 0 & 1 & 2x_d+6b & 3(x_d+3b)^2 & 4(x_d+3b)^3 & 5(x_d+3b)^4 \\ 0 & 0 & 2 & 6x_d+18b & 12(x_d+3b)^2 & 20(x_d+3b)^3 \\ 0 & 0 & 2 & 6x_d+24b & 12(x_d+4b)^2 & 20(x_d+4b)^3 \end{pmatrix} \quad (16)$$

The inverse of the matrix $Z = J^{-1}$ is computed using Maple 18 from which the continuous scheme is obtained using (4) and evaluating it at $x = x_{d+3}, x = x_{d+4}$ and its derivatives at $x = x_{d+1}$ and $x = x_{d+2}$ the following discrete schemes are obtained

$$y_{d+1} = \frac{463}{1282}b^2g_{d+3} - \frac{127}{2564}b^2q_{d+4} - \frac{5239}{5128}bf_{d+1} - \frac{2081}{5128}bf_{d+3} - \frac{137}{641}y_d + \frac{778}{641}y_{d+2}$$

$$y_{d+2} = \frac{3552}{5975}b^2g_{d+3} - \frac{374}{5975}b^2q_{d+4} + \frac{10478}{5975}bf_{d+2} - \frac{5168}{5975}bf_{d+3} - \frac{133}{1195}y_d + \frac{1328}{1195}y_{d+1}$$

$$y_{d+3} = \frac{136}{5239}y_d - \frac{1161}{5239}y_{d+1} + \frac{6264}{5239}y_{d+2} + \frac{4350}{5239}bf_{d+3} - \frac{1530}{5239}b^2g_{d+3} + \frac{108}{5239}b^2q_{d+4}$$

$$y_{d+4} = \frac{249}{5239}y_d - \frac{1856}{5239}y_{d+1} + \frac{6846}{5239}y_{d+2} + \frac{9120}{5239}bf_{d+3} + \frac{1128}{5239}b^2g_{d+3} + \frac{660}{5239}b^2q_{d+4} \quad (17)$$

D. Construction of Extended Second Derivative Block Backward Differentiation Formulae Method for k = 4

Here, we incorporate one extrapolated future point at $x = x_{d+5}$ as a collocation point, thus the interpolation points, $g = 4$ and the collocation points $l = 3$ are considered,

Therefore, (4) becomes

$$y(x) = \alpha_0(x)y_d + \alpha_1(x)y_{d+1} + \alpha_2(x)y_{d+2} + \alpha_3(x)y_{d+3} + b\beta_4(x)f_{d+4} + b^2\gamma_4(x)g_{d+4} + b^2\delta_5(x)q_{d+5} \quad (18)$$

The matrix J in (9) becomes

$$J=\begin{pmatrix} 1 & x_d & x_d^2 & x_d^3 & x_d^4 & x_d^5 & x_d^6 \\ 1 & x_d+b & (x_d+b)^2 & (x_d+b)^3 & (x_d+b)^4 & (x_d+b)^5 & (x_d+b)^6 \\ 1 & x_d+2b & (x_d+2b)^2 & (x_d+2b)^3 & (x_d+2b)^4 & (x_d+2b)^5 & (x_d+2b)^6 \\ 1 & x_d+3b & (x_d+3b)^2 & (x_d+3b)^3 & (x_d+3b)^4 & (x_d+3b)^5 & (x_d+3b)^6 \\ 0 & 1 & 2x_d+8b & 3(x_d+4b)^2 & 4(x_d+4b)^3 & 5(x_d+4b)^4 & 6(x_d+4b)^5 \\ 0 & 0 & 2 & 6x_d+24b & 12(x_d+4b)^2 & 20(x_d+4b)^3 & 30(x_d+4b)^4 \\ 0 & 0 & 2 & 6x_d+30b & 12(x_d+5b)^2 & 20(x_d+5b)^3 & 30(x_d+5b)^4 \end{pmatrix} \quad (19)$$

The inverse of the matrix $Z = J^{-1}$ is computed using Maple 18 from which the continuous scheme is obtained using (4) and evaluating it at $x = x_{d+4}, x = x_{d+5}$ and its derivatives at $x = x_{d+1}, x = x_{d+2}$ and $x = x_{d+3}$ the following discrete schemes are obtained

$$y_{d+1} = -\frac{5758}{25407}b^2g_{d+4} + \frac{118}{4705}b^2q_{d+5} - \frac{299618}{381105}bf_{d+1} + \frac{105758}{381105}bf_{d+4} - \frac{5561}{42345}y_d + \frac{26611}{14115}y_{d+2} - \frac{31927}{42345}y_{d+3}$$

$$y_{d+2} = \frac{50030}{152373}b^2g_{d+4} - \frac{4808}{152373}b^2q_{d+5} - \frac{299618}{152373}bf_{d+2} - \frac{68372}{152373}bf_{d+4} + \frac{30671}{457119}y_d - \frac{41048}{50791}y_{d+1} + \frac{795880}{457119}y_{d+3}$$

$$y_{d+3} = \frac{209124}{667619}b^2g_{d+4} - \frac{15462}{667619}b^2q_{d+5} + \frac{898854}{667619}bf_{d+3} - \frac{356724}{667619}bf_{d+4} + \frac{17486}{667619}y_d - \frac{160461}{667619}y_{d+1} + \frac{810594}{667619}y_{d+3}$$

$$y_{d+4} = -\frac{1431}{149809}y_d + \frac{12352}{149809}y_{d+1} - \frac{52920}{149809}y_{d+2} + \frac{191808}{149809}y_{d+3} + \frac{117300}{149809}bf_{d+4} - \frac{35928}{149809}b^2g_{d+4} + \frac{1728}{149809}b^2q_{d+5}$$

$$y_{d+5} = -\frac{3506}{149809}y_d + \frac{27855}{149809}y_{d+1} - \frac{101390}{149809}y_{d+2} + \frac{226850}{149809}y_{d+3} + \frac{243420}{149809}bf_{d+4} + \frac{51420}{149809}b^2g_{d+4} + \frac{15540}{149809}b^2q_{d+5} \quad (20)$$

III. CONVERGENCE ANALYSIS

Here, the investigations of order, error constant, consistency, zero stability and region of the absolute stability of (14), (17) and (20) are carried out.

E. Order and Error Constant

The order and error constants for (14) are obtained as follows

$$C_0 = C_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ but } C_2 = \begin{pmatrix} \frac{7}{5} \\ -\frac{10}{29} \\ \frac{6}{29} \end{pmatrix}$$

Therefore, (14) has order $p=1$ and error constants, $\left(\frac{7}{5}, -\frac{10}{29}, \frac{6}{29}\right)^T$

With the same approach, (17) and can be presented as

$$C_0 = C_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ but } C_2 = \begin{pmatrix} \frac{799}{2564} \\ \frac{3178}{5975} \\ -\frac{1422}{5239} \\ \frac{1788}{5239} \end{pmatrix}$$

Therefore, (17) has order $p=1$ and error constants, $\left(\frac{799}{2564}, \frac{3178}{5975}, -\frac{1422}{5239}, \frac{1788}{5239}\right)^T$

Applying the same approach, (20) and can be obtained as

$$C_0 = C_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ but } C_2 = \begin{pmatrix} -\frac{25604}{127035} \\ \frac{15074}{50791} \\ \frac{193662}{667619} \\ -\frac{34200}{149809} \\ -\frac{3611305}{299618} \end{pmatrix}$$

Therefore, (20) has order $p=1$ and error constants, $\left(-\frac{25604}{127035}, \frac{15074}{50791}, \frac{193662}{667619}, -\frac{34200}{149809}, -\frac{3611305}{299618}\right)^T$

F. Consistency

Since $p=1$ in (14), (17) and (20) satisfying the condition for consistency of order $p \geq 1$, then the schemes are consistent.

G. Zero Stability

The zero stability for (14) is determined as follows

$$\begin{pmatrix} 1 & 0 & 0 \\ -\frac{32}{29} & 1 & 0 \\ -\frac{33}{29} & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{d+1} \\ y_{d+2} \\ y_{d+3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & \frac{3}{29} \\ 0 & 0 & \frac{4}{29} \end{pmatrix} \begin{pmatrix} y_{d-2} \\ y_{d-1} \\ y_d \end{pmatrix}$$

$$+b \begin{pmatrix} \frac{29}{10} & -\frac{19}{10} & 0 \\ 0 & \frac{26}{29} & 0 \\ 0 & \frac{54}{29} & 0 \end{pmatrix} \begin{pmatrix} f_{d+1} \\ f_{d+2} \\ f_{d+3} \end{pmatrix} + b \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_{d-2} \\ f_{d-1} \\ f_d \end{pmatrix} + b^2 \begin{pmatrix} 0 & \frac{101}{60} & 0 \\ 0 & -\frac{34}{87} & 0 \\ 0 & \frac{1}{29} & 0 \end{pmatrix} \begin{pmatrix} g_{d+1} \\ g_{d+2} \\ g_{d+3} \end{pmatrix}$$

$$+b^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} g_{d-2} \\ g_{d-1} \\ g_d \end{pmatrix} + b^2 \begin{pmatrix} 0 & -\frac{17}{60} & 0 \\ 0 & \frac{4}{87} & 0 \\ 0 & \frac{5}{29} & 0 \end{pmatrix} \begin{pmatrix} q_{d+1} \\ q_{d+2} \\ q_{d+3} \end{pmatrix} + b^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_{d-2} \\ q_{d-1} \\ q_d \end{pmatrix}$$

where $B_2^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{32}{29} & 1 & 0 \\ -\frac{33}{29} & 0 & 1 \end{pmatrix}, B_1^{(1)} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & \frac{3}{29} \\ 0 & 0 & \frac{4}{29} \end{pmatrix}, D_2^{(1)} = \begin{pmatrix} \frac{29}{10} & -\frac{19}{10} & 0 \\ 0 & \frac{26}{29} & 0 \\ 0 & \frac{54}{29} & 0 \end{pmatrix}, E_2^{(1)} = \begin{pmatrix} 0 & \frac{101}{60} & 0 \\ 0 & -\frac{34}{87} & 0 \\ 0 & \frac{1}{29} & 0 \end{pmatrix}$

and $L_2^{(1)} = \begin{pmatrix} 0 & -\frac{17}{60} & 0 \\ 0 & \frac{4}{87} & 0 \\ 0 & \frac{5}{29} & 0 \end{pmatrix}$

$$\begin{aligned} \mathrm{H}(R) &= \det\left(RB_2^{(1)} - B_1^{(1)}\right) \\ &= \left|RB_2^{(1)} - B_1^{(1)}\right| = 0. \end{aligned} \qquad (21)$$

Now we have,

$$H(R) = \left| R \begin{pmatrix} 1 & 0 & 0 \\ -\frac{32}{29} & 1 & 0 \\ -\frac{33}{29} & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & \frac{3}{29} \\ 0 & 0 & \frac{4}{29} \end{pmatrix} \right| = \left| \begin{pmatrix} R & 0 & 0 \\ -\frac{32}{29}R & R & 0 \\ -\frac{33}{29}R & 0 & R \end{pmatrix} - \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & \frac{3}{29} \\ 0 & 0 & \frac{4}{29} \end{pmatrix} \right|$$

$$\Rightarrow H(R) = \begin{vmatrix} R & 0 & 1 \\ -\frac{32}{29}R & R & -\frac{3}{29} \\ -\frac{33}{29}R & 0 & R - \frac{4}{29} \end{vmatrix}$$

The following are obtained using Maple (18) software,

$$H(R) = R^3 + R^2$$

$$\Rightarrow R^3 + R^2 = 0$$

$\Rightarrow R_1=-1, R_2=0, R_3=0$ .Since $|R_i|<1,\ i=1,2,3$, (14) is zero stable.

By the same procedure for (17)

$$\begin{pmatrix} 1 & -\frac{778}{641} & 0 & 0 \\ -\frac{1328}{1195} & 1 & 0 & 0 \\ \frac{1161}{5239} & -\frac{6264}{5239} & 1 & 0 \\ \frac{1856}{5239} & -\frac{6846}{5239} & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{d+1} \\ y_{d+2} \\ y_{d+3} \\ y_{d+4} \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & \frac{137}{641} \\ 0 & 0 & 0 & \frac{133}{1195} \\ 0 & 0 & 0 & -\frac{136}{5239} \\ 0 & 0 & 0 & -\frac{249}{5239} \end{pmatrix}\begin{pmatrix} y_{d-3} \\ y_{d-2} \\ y_{d-1} \\ y_{d} \end{pmatrix}$$

$$+b\begin{pmatrix} -\frac{5239}{5128} & 0 & -\frac{2081}{5128} & 0 \\ 0 & \frac{10478}{5975} & -\frac{5168}{5975} & 0 \\ 0 & 0 & \frac{4350}{5239} & 0 \\ 0 & 0 & \frac{9120}{5239} & 0 \end{pmatrix}\begin{pmatrix} f_{d+1} \\ f_{d+2} \\ f_{d+3} \\ f_{d+4} \end{pmatrix}+b\begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}\begin{pmatrix} f_{d-3} \\ f_{d-2} \\ f_{d-1} \\ f_{d} \end{pmatrix}+b^2\begin{pmatrix} 0 & 0 & \frac{463}{1282} & 0 \\ 0 & 0 & \frac{3552}{5975} & 0 \\ 0 & 0 & -\frac{1530}{5239} & 0 \\ 0 & 0 & \frac{1128}{5239} & 0 \end{pmatrix}\begin{pmatrix} g_{d+1} \\ g_{d+2} \\ g_{d+3} \\ g_{d+4} \end{pmatrix}$$

$$+b^2\begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}\begin{pmatrix} g_{d-3} \\ g_{d-2} \\ g_{d-1} \\ g_{d} \end{pmatrix}+b^2\begin{pmatrix} 0 & 0 & 0 & -\frac{127}{2564} \\ 0 & 0 & 0 & -\frac{374}{5975} \\ 0 & 0 & 0 & \frac{108}{5239} \\ 0 & 0 & 0 & \frac{660}{5239} \end{pmatrix}\begin{pmatrix} q_{d+1} \\ q_{d+2} \\ q_{d+3} \\ q_{d+4} \end{pmatrix}+b^2\begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}\begin{pmatrix} q_{d-3} \\ q_{d-2} \\ q_{d-1} \\ q_{d} \end{pmatrix}$$

where

$$B_2^{(2)}=\begin{pmatrix} 1 & -\frac{778}{641} & 0 & 0 \\ -\frac{1328}{1195} & 1 & 0 & 0 \\ \frac{1161}{5239} & -\frac{6264}{5239} & 1 & 0 \\ \frac{1856}{5239} & -\frac{6846}{5239} & 0 & 1 \end{pmatrix}, B_1^{(2)}=\begin{pmatrix} 0 & 0 & 0 & \frac{137}{641} \\ 0 & 0 & 0 & \frac{133}{1195} \\ 0 & 0 & 0 & -\frac{136}{5239} \\ 0 & 0 & 0 & -\frac{249}{5239} \end{pmatrix}, D_2^{(2)}=\begin{pmatrix} -\frac{5239}{5128} & 0 & -\frac{2081}{5128} & 0 \\ 0 & \frac{10478}{5975} & -\frac{5168}{5975} & 0 \\ 0 & 0 & \frac{4350}{5239} & 0 \\ 0 & 0 & \frac{9120}{5239} & 0 \end{pmatrix}$$

$$E_2^{(2)}=\begin{pmatrix} 0 & 0 & \frac{463}{1282} & 0 \\ 0 & 0 & \frac{3552}{5975} & 0 \\ 0 & 0 & -\frac{1530}{5239} & 0 \\ 0 & 0 & \frac{1128}{5239} & 0 \end{pmatrix} \text{ and } L_2^{(2)}=\begin{pmatrix} 0 & 0 & 0 & -\frac{127}{2564} \\ 0 & 0 & 0 & -\frac{374}{5975} \\ 0 & 0 & 0 & \frac{108}{5239} \\ 0 & 0 & 0 & \frac{660}{5239} \end{pmatrix}$$

$$H(R)=\det\left(RB_2^{(2)}-B_1^{(2)}\right) = \left|RB_2^{(2)}-B_1^{(2)}\right|=0. \qquad (22)$$

Now we have,

$$H(R)=\left|R\begin{pmatrix} 1 & -\frac{778}{641} & 0 & 0 \\ -\frac{1328}{1195} & 1 & 0 & 0 \\ \frac{1161}{5239} & -\frac{6264}{5239} & 1 & 0 \\ \frac{1856}{5239} & -\frac{6846}{5239} & 0 & 1 \end{pmatrix}-\begin{pmatrix} 0 & 0 & 0 & \frac{137}{641} \\ 0 & 0 & 0 & \frac{133}{1195} \\ 0 & 0 & 0 & -\frac{136}{5239} \\ 0 & 0 & 0 & -\frac{249}{5239} \end{pmatrix}\right|$$

$$=\left|\begin{pmatrix} R & -\frac{778}{641}R & 0 & 0 \\ -\frac{1328}{1195}R & R & 0 & 0 \\ \frac{1161}{5239}R & -\frac{6264}{5239}R & R & 0 \\ \frac{1856}{5239}R & -\frac{6846}{5239}R & 0 & R \end{pmatrix}-\begin{pmatrix} 0 & 0 & 0 & \frac{137}{641} \\ 0 & 0 & 0 & \frac{133}{1195} \\ 0 & 0 & 0 & -\frac{136}{5239} \\ 0 & 0 & 0 & -\frac{249}{5239} \end{pmatrix}\right|$$

$$\Rightarrow H(R)=\begin{vmatrix} R & -\frac{778}{641}R & 0 & -\frac{137}{641} \\ -\frac{1328}{1195}R & R & 0 & -\frac{133}{1195} \\ \frac{1161}{5239}R & -\frac{6264}{5239}R & R & \frac{136}{5239} \\ \frac{1856}{5239}R & -\frac{6846}{5239}R & 0 & R+\frac{249}{5239} \end{vmatrix}$$

Using Maple (18) software, we obtain:

$$H(R)=-\frac{267189}{765995}R^4-\frac{267189}{765995}R^3$$

$$\Rightarrow -\frac{267189}{765995}R^4-\frac{267189}{765995}R^3=0$$

$$\Rightarrow R_1=-1, R_2=0, R_3=0, R_4=0$$

Since $|R_i|<1,\ i=1,2,3,4$, (17) is zero stable.

Following the same procedure for (20)

$$\begin{pmatrix} 1 & -\frac{26611}{14115} & \frac{31927}{42345} & 0 & 0 \\ \frac{41048}{50791} & 1 & -\frac{795880}{457119} & 0 & 0 \\ \frac{160461}{667619} & -\frac{810594}{667619} & 1 & 0 & 0 \\ -\frac{12352}{149809} & \frac{52920}{149809} & -\frac{191808}{149809} & 1 & 0 \\ -\frac{27855}{149809} & \frac{101390}{149809} & -\frac{226850}{149809} & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{d+1} \\ y_{d+2} \\ y_{d+3} \\ y_{d+4} \\ y_{d+5} \end{pmatrix}=\begin{pmatrix} 0&0&0&0&\frac{5561}{42345} \\ 0&0&0&0&-\frac{30671}{457119} \\ 0&0&0&0&-\frac{17486}{667619} \\ 0&0&0&0&\frac{1431}{149809} \\ 0&0&0&0&\frac{3506}{149809} \end{pmatrix}\begin{pmatrix} y_{d-4} \\ y_{d-3} \\ y_{d-2} \\ y_{d-1} \\ y_{d} \end{pmatrix}$$

$$+b\begin{pmatrix} -\frac{299618}{381105} & 0 & 0 & \frac{105758}{381105} & 0 \\ 0 & -\frac{299618}{152373} & 0 & -\frac{68372}{152373} & 0 \\ 0 & 0 & \frac{898854}{667619} & -\frac{356724}{667619} & 0 \\ 0 & 0 & 0 & \frac{117300}{149809} & 0 \\ 0 & 0 & 0 & \frac{243420}{149809} & 0 \end{pmatrix}\begin{pmatrix} f_{d+1} \\ f_{d+2} \\ f_{d+3} \\ f_{d+4} \\ f_{d+5} \end{pmatrix}+b\begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}\begin{pmatrix} f_{d-4} \\ f_{d-3} \\ f_{d-2} \\ f_{d-1} \\ f_{d} \end{pmatrix}$$

$$+b^2\begin{pmatrix} 0 & 0 & 0 & -\frac{5758}{25407} & 0 \\ 0 & 0 & 0 & \frac{50030}{152373} & 0 \\ 0 & 0 & 0 & \frac{209124}{667619} & 0 \\ 0 & 0 & 0 & -\frac{35928}{149809} & 0 \\ 0 & 0 & 0 & \frac{51420}{149809} & 0 \end{pmatrix}\begin{pmatrix} g_{d+1} \\ g_{d+2} \\ g_{d+3} \\ g_{d+4} \\ g_{d+5} \end{pmatrix}+b^2\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} g_{d-4} \\ g_{d-3} \\ g_{d-2} \\ g_{d-1} \\ g_{d} \end{pmatrix}$$

$$+b^2\begin{pmatrix} 0 & 0 & 0 & \frac{118}{4705} & 0 \\ 0 & 0 & 0 & -\frac{4808}{152373} & 0 \\ 0 & 0 & 0 & -\frac{15462}{667619} & 0 \\ 0 & 0 & 0 & \frac{1728}{149809} & 0 \\ 0 & 0 & 0 & \frac{15540}{149809} & 0 \end{pmatrix}\begin{pmatrix} q_{d+1} \\ q_{d+2} \\ q_{d+3} \\ q_{d+4} \\ q_{d+5} \end{pmatrix}+b^2\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} q_{d-4} \\ q_{d-3} \\ q_{d-2} \\ q_{d-1} \\ q_{d} \end{pmatrix}$$

where

$$B_2^{(3)}=\begin{pmatrix} 1 & -\frac{26611}{14115} & \frac{31927}{42345} & 0 & 0 \\ \frac{41048}{50791} & 1 & -\frac{795880}{457119} & 0 & 0 \\ \frac{160461}{667619} & -\frac{810594}{667619} & 1 & 0 & 0 \\ -\frac{12352}{149809} & \frac{52920}{149809} & -\frac{191808}{149809} & 1 & 0 \\ -\frac{27855}{149809} & \frac{101390}{149809} & -\frac{226850}{149809} & 0 & 1 \end{pmatrix}, B_1^{(3)}=\begin{pmatrix} 0 & 0 & 0 & 0 & \frac{5561}{42345} \\ 0 & 0 & 0 & 0 & -\frac{30671}{457119} \\ 0 & 0 & 0 & 0 & -\frac{17486}{667619} \\ 0 & 0 & 0 & 0 & \frac{1431}{149809} \\ 0 & 0 & 0 & 0 & \frac{3506}{149809} \end{pmatrix}, D_2^{(3)}=\begin{pmatrix} -\frac{299618}{381105} & 0 & 0 & \frac{105758}{381105} & 0 \\ 0 & -\frac{299618}{152373} & 0 & -\frac{68372}{152373} & 0 \\ 0 & 0 & \frac{898854}{667619} & -\frac{356724}{667619} & 0 \\ 0 & 0 & 0 & \frac{117300}{149809} & 0 \\ 0 & 0 & 0 & \frac{243420}{149809} & 0 \end{pmatrix},$$

$$E_2^{(3)}=\begin{pmatrix} 0 & 0 & 0 & -\frac{5758}{25407} & 0 \\ 0 & 0 & 0 & \frac{50030}{152373} & 0 \\ 0 & 0 & 0 & \frac{209124}{667619} & 0 \\ 0 & 0 & 0 & -\frac{35928}{149809} & 0 \\ 0 & 0 & 0 & \frac{51420}{149809} & 0 \end{pmatrix} \text{ and } L_2^{(3)}=\begin{pmatrix} 0 & 0 & 0 & \frac{118}{4705} & 0 \\ 0 & 0 & 0 & -\frac{4808}{152373} & 0 \\ 0 & 0 & 0 & -\frac{15462}{667619} & 0 \\ 0 & 0 & 0 & \frac{1728}{149809} & 0 \\ 0 & 0 & 0 & \frac{15540}{149809} & 0 \end{pmatrix}$$

$$H(R)=\det\left(RB_2^{(3)}-B_1^{(3)}\right) = \left|RB_2^{(3)}-B_1^{(3)}\right|=0. \qquad (23)$$

Now we have,

$$H(R)=\left|R\begin{pmatrix} 1 & -\frac{26611}{14115} & \frac{31927}{42345} & 0 & 0 \\ \frac{41048}{50791} & 1 & -\frac{795880}{457119} & 0 & 0 \\ \frac{160461}{667619} & -\frac{810594}{667619} & 1 & 0 & 0 \\ -\frac{12352}{149809} & \frac{52920}{149809} & -\frac{191808}{149809} & 1 & 0 \\ -\frac{27855}{149809} & \frac{101390}{149809} & -\frac{226850}{149809} & 0 & 1 \end{pmatrix}-\begin{pmatrix} 0 & 0 & 0 & 0 & \frac{5561}{42345} \\ 0 & 0 & 0 & 0 & -\frac{30671}{457119} \\ 0 & 0 & 0 & 0 & -\frac{17486}{667619} \\ 0 & 0 & 0 & 0 & \frac{1431}{149809} \\ 0 & 0 & 0 & 0 & \frac{3506}{149809} \end{pmatrix}\right|$$

$$=\left|\begin{pmatrix} R & -\frac{26611}{14115} & \frac{31927}{42345}R & 0 & 0 \\ \frac{41048}{50791}R & R & -\frac{795880}{457119}R & 0 & 0 \\ \frac{160461}{667619}R & -\frac{810594}{667619}R & R & 0 & 0 \\ -\frac{12352}{149809}R & \frac{52920}{149809}R & -\frac{191808}{149809}R & R & 0 \\ -\frac{27855}{149809}R & \frac{101390}{149809}R & -\frac{226850}{149809}R & 0 & R \end{pmatrix}-\begin{pmatrix} 0 & 0 & 0 & 0 & \frac{5561}{42345} \\ 0 & 0 & 0 & 0 & -\frac{30671}{457119} \\ 0 & 0 & 0 & 0 & -\frac{17486}{667619} \\ 0 & 0 & 0 & 0 & \frac{1431}{149809} \\ 0 & 0 & 0 & 0 & \frac{3506}{149809} \end{pmatrix}\right|$$

$$\Rightarrow H(R)=\begin{vmatrix} R & -\frac{26611}{14115} & \frac{31927}{42345}R & 0 & -\frac{5561}{42345} \\ \frac{41048}{50791}R & R & -\frac{795880}{457119}R & 0 & \frac{30671}{457119} \\ \frac{160461}{667619}R & -\frac{810594}{667619}R & R & 0 & \frac{17486}{667619} \\ -\frac{12352}{149809}R & \frac{52920}{149809}R & -\frac{191808}{149809}R & R & -\frac{1431}{149809} \\ -\frac{27855}{149809}R & \frac{101390}{149809}R & -\frac{226850}{149809}R & 0 & R-\frac{3506}{149809} \end{vmatrix}$$

Using Maple (18) software, we obtain:

$$H(R)=\frac{26572199993504}{95725210403667}R^5+\frac{26572199993504}{95725210403667}R^4$$

$$\Rightarrow \frac{26572199993504}{95725210403667}R^5+\frac{26572199993504}{95725210403667}R^4=0$$

$$\Rightarrow R_1=-1, R_2=0, R_3=0, R_4=0, R_5=0$$

Since $|R_i|<1,\ i=1,2,3,4,5$, (20) is zero stable.

H. Convergence

Since (14), (17) and (20) are both consistent and zero stable, therefore they are convergent.

I. Region of Absolute Stability

The regions of absolute stability of the numerical methods for DDEs are considered. We considered finding the Z - and W -stability by applying (14), (17) and (20) to the DDEs of this form:

$$x'(t)=ax(t)+cx(t-\tau),\ t\geq t_0 \qquad (24)$$
$$x(t)=m(t),\qquad t\leq t_0$$

where $m(t)$ is the initial function, a,c are complex coefficients, $\tau=bs, s\in Z^+$, b is the step size and

$b=\frac{\tau}{s}, s$ is a positive integer. Let $M_1 = ba$ and $M_2 = bc$, then from (14), let

$$Y_{D+3}=\begin{pmatrix} y_{d+1}\\ y_{d+2}\\ y_{d+3}\end{pmatrix}, Y_D=\begin{pmatrix} y_{d-2}\\ y_{d-1}\\ y_{d}\end{pmatrix}, F_{D+3}=\begin{pmatrix} f_{d+1}\\ f_{d+2}\\ f_{d+3}\end{pmatrix}, F_D=\begin{pmatrix} f_{d-2}\\ f_{d-1}\\ f_{d}\end{pmatrix}, G_{D+3}=\begin{pmatrix} g_{d+1}\\ g_{d+2}\\ g_{d+3}\end{pmatrix}, G_D=\begin{pmatrix} g_{d-2}\\ g_{d-1}\\ g_{d}\end{pmatrix},$$

$$Q_{D+3}=\begin{pmatrix} q_{d+1}\\ q_{d+2}\\ q_{d+3}\end{pmatrix} \text{ and } Q_D=\begin{pmatrix} q_{d-2}\\ q_{d-1}\\ q_{d}\end{pmatrix}$$

Since

$$B_2^{(1)}=\begin{pmatrix} 1 & 0 & 0\\ -\frac{32}{29} & 1 & 0\\ -\frac{33}{29} & 0 & 1\end{pmatrix}, B_1^{(1)}=\begin{pmatrix} 0 & 0 & -1\\ 0 & 0 & \frac{3}{29}\\ 0 & 0 & \frac{4}{29}\end{pmatrix}, D_2^{(1)}=\begin{pmatrix} \frac{29}{10} & -\frac{19}{10} & 0\\ 0 & \frac{26}{29} & 0\\ 0 & \frac{54}{29} & 0\end{pmatrix}, E_2^{(1)}=\begin{pmatrix} 0 & \frac{101}{60} & 0\\ 0 & -\frac{34}{87} & 0\\ 0 & \frac{1}{29} & 0\end{pmatrix}$$

and $L_2^{(1)}=\begin{pmatrix} 0 & -\frac{17}{60} & 0\\ 0 & \frac{4}{87} & 0\\ 0 & \frac{5}{29} & 0\end{pmatrix}$

we have,

$$B_2^{(1)}Y_{D+2}=B_1^{(1)}Y_{D+1}-b\left(\sum_{u=1}^{2}\left(D_u^{(1)}F_{D+u}+bE_u^{(1)}G_{D+u}+bL_u^{(1)}Q_{D+u}\right)\right) \quad (25)$$

Applying the same approach for (17), we have

$$Y_{D+4}=\begin{pmatrix} y_{d+1}\\ y_{d+2}\\ y_{d+3}\\ y_{d+4}\end{pmatrix}, Y_D=\begin{pmatrix} y_{d-3}\\ y_{d-2}\\ y_{d-1}\\ y_{d}\end{pmatrix}, F_{D+4}=\begin{pmatrix} f_{d+1}\\ f_{d+2}\\ f_{d+3}\\ f_{d+4}\end{pmatrix}, F_D=\begin{pmatrix} f_{d-3}\\ f_{d-2}\\ f_{d-1}\\ f_{d}\end{pmatrix}, G_{D+4}=\begin{pmatrix} g_{d+1}\\ g_{d+2}\\ g_{d+3}\\ g_{d+4}\end{pmatrix}, G_D=\begin{pmatrix} g_{d-3}\\ g_{d-2}\\ g_{d-1}\\ g_{d}\end{pmatrix}$$

$$Q_{D+4}=\begin{pmatrix} q_{d+1}\\ q_{d+2}\\ q_{d+3}\\ q_{d+4}\end{pmatrix} \text{ and } Q_D=\begin{pmatrix} q_{d-3}\\ q_{d-2}\\ q_{d-1}\\ q_{d}\end{pmatrix}$$

Since

$$B_2^{(2)}=\begin{pmatrix} 1 & -\frac{778}{641} & 0 & 0\\ -\frac{1328}{1195} & 1 & 0 & 0\\ \frac{1161}{5239} & -\frac{6264}{5239} & 1 & 0\\ \frac{1856}{5239} & -\frac{6846}{5239} & 0 & 1\end{pmatrix}, B_1^{(2)}=\begin{pmatrix} 0 & 0 & 0 & \frac{137}{641}\\ 0 & 0 & 0 & \frac{133}{1195}\\ 0 & 0 & 0 & -\frac{136}{5239}\\ 0 & 0 & 0 & -\frac{249}{5239}\end{pmatrix}, D_2^{(2)}=\begin{pmatrix} -\frac{5239}{5128} & 0 & -\frac{2081}{5128} & 0\\ 0 & \frac{10478}{5975} & -\frac{5168}{5975} & 0\\ 0 & 0 & \frac{4350}{5239} & 0\\ 0 & 0 & \frac{9120}{5239} & 0\end{pmatrix}$$

$$E_2^{(2)}=\begin{pmatrix} 0 & 0 & \frac{463}{1282} & 0\\ 0 & 0 & \frac{3552}{5975} & 0\\ 0 & 0 & -\frac{1530}{5239} & 0\\ 0 & 0 & \frac{1128}{5239} & 0\end{pmatrix} \text{ and } L_2^{(2)}=\begin{pmatrix} 0 & 0 & 0 & -\frac{127}{2564}\\ 0 & 0 & 0 & -\frac{374}{5975}\\ 0 & 0 & 0 & \frac{108}{5239}\\ 0 & 0 & 0 & \frac{660}{5239}\end{pmatrix}$$

we have,

$$B_2^{(2)}Y_{D+2}=B_1^{(2)}Y_{D+1}-b\left(\sum_{u=1}^{2}\left(D_u^{(2)}F_{D+u}+bE_u^{(2)}G_{D+u}+bL_u^{(2)}Q_{D+u}\right)\right) \quad (26)$$

Applying the same approach for (20), we have

$$Y_{D+5}=\begin{pmatrix} y_{d+1}\\ y_{d+2}\\ y_{d+3}\\ y_{d+4}\\ y_{d+5}\end{pmatrix}, Y_D=\begin{pmatrix} y_{d-4}\\ y_{d-3}\\ y_{d-2}\\ y_{d-1}\\ y_{d}\end{pmatrix}, F_{D+5}=\begin{pmatrix} f_{d+1}\\ f_{d+2}\\ f_{d+3}\\ f_{d+4}\\ f_{d+5}\end{pmatrix}, F_D=\begin{pmatrix} f_{d-4}\\ f_{d-3}\\ f_{d-2}\\ f_{d-1}\\ f_{d}\end{pmatrix}, G_{D+5}=\begin{pmatrix} g_{d+1}\\ g_{d+2}\\ g_{d+3}\\ g_{d+4}\\ g_{d+5}\end{pmatrix}, G_D=\begin{pmatrix} g_{d-4}\\ g_{d-3}\\ g_{d-2}\\ g_{d-1}\\ g_{d}\end{pmatrix}$$

$$Q_{D+5}=\begin{pmatrix} q_{d+1}\\ q_{d+2}\\ q_{d+3}\\ q_{d+4}\\ q_{d+5}\end{pmatrix} \text{ and } Q_D=\begin{pmatrix} q_{d-4}\\ q_{d-3}\\ q_{d-2}\\ q_{d-1}\\ q_{d}\end{pmatrix}$$

Since

$$B_2^{(3)}=\begin{pmatrix} 1 & -\frac{26611}{14115} & \frac{31927}{42345} & 0 & 0\\ \frac{41048}{50791} & 1 & -\frac{795880}{457119} & 0 & 0\\ \frac{160461}{667619} & -\frac{810594}{667619} & 1 & 0 & 0\\ -\frac{12352}{149809} & \frac{52920}{149809} & -\frac{191808}{149809} & 1 & 0\\ -\frac{27855}{149809} & \frac{101390}{149809} & -\frac{226850}{149809} & 0 & 1\end{pmatrix}, B_1^{(3)}=\begin{pmatrix} 0 & 0 & 0 & 0 & \frac{5561}{42345}\\ 0 & 0 & 0 & 0 & -\frac{30671}{457119}\\ 0 & 0 & 0 & 0 & -\frac{17486}{667619}\\ 0 & 0 & 0 & 0 & \frac{1431}{149809}\\ 0 & 0 & 0 & 0 & \frac{3506}{149809}\end{pmatrix}, D_2^{(3)}=\begin{pmatrix} -\frac{299618}{381105} & 0 & 0 & \frac{105758}{381105} & 0\\ 0 & -\frac{299618}{152373} & 0 & -\frac{68372}{152373} & 0\\ 0 & 0 & \frac{898854}{667619} & -\frac{356724}{667619} & 0\\ 0 & 0 & 0 & \frac{117300}{149809} & 0\\ 0 & 0 & 0 & \frac{243420}{149809} & 0\end{pmatrix},$$

$$E_2^{(3)}=\begin{pmatrix} 0 & 0 & 0 & -\frac{5758}{25407} & 0\\ 0 & 0 & 0 & \frac{50030}{152373} & 0\\ 0 & 0 & 0 & \frac{209124}{667619} & 0\\ 0 & 0 & 0 & -\frac{35928}{149809} & 0\\ 0 & 0 & 0 & \frac{51420}{149809} & 0\end{pmatrix} \text{ and } L_2^{(3)}=\begin{pmatrix} 0 & 0 & 0 & \frac{118}{4705} & 0\\ 0 & 0 & 0 & -\frac{4808}{152373} & 0\\ 0 & 0 & 0 & -\frac{15462}{667619} & 0\\ 0 & 0 & 0 & \frac{1728}{149809} & 0\\ 0 & 0 & 0 & \frac{15540}{149809} & 0\end{pmatrix}$$

we have,

$$B_2^{(3)}Y_{D+2}=B_1^{(3)}Y_{D+1}-b\left(\sum_{u=1}^{2}\left(D_u^{(3)}F_{D+u}+bE_u^{(3)}G_{D+u}+bL_u^{(3)}Q_{D+u}\right)\right) \quad (27)$$

The polynomials of Z - and W -stability are constructed by applying (25), (26) and (27) to (24) and (14), (17) and (20) to (24) as presented below

$$P^{(1)}(\phi)=\det\begin{bmatrix}\left(B_2^{(1)}-M_1\left(D_2^{(1)}-E_2^{(1)}\right)\right)\phi^{2+s}\\ -\left(B_1^{(1)}-M_1\left(D_1^{(1)}-E_1^{(1)}\right)\right)\phi^{1+s}-M_2\left(\sum_{u=1}^{2}\left(D_u^{(1)}-E_u^{(1)}-L_u^{(1)}\right)\phi^u\right)\end{bmatrix} \quad (28)$$

$$P^{(2)}(\phi)=\det\begin{bmatrix}\left(B_2^{(2)}-M_1\left(D_2^{(2)}-E_2^{(2)}\right)\right)\phi^{2+s}\\ -\left(B_1^{(2)}-M_1\left(D_1^{(2)}-E_1^{(2)}\right)\right)\phi^{1+s}-M_2\left(\sum_{u=1}^{2}\left(D_u^{(2)}-E_u^{(2)}-L_u^{(2)}\right)\phi^u\right)\end{bmatrix} \quad (29)$$

$$P^{(3)}(\phi)=\det\begin{bmatrix}\left(B_2^{(3)}-M_1\left(D_2^{(3)}-E_2^{(3)}\right)\right)\phi^{2+s}\\ -\left(B_1^{(3)}-M_1\left(D_1^{(3)}-E_1^{(3)}\right)\right)\phi^{1+s}-M_2\left(\sum_{u=1}^{2}\left(D_u^{(3)}-E_u^{(3)}-L_u^{(3)}\right)\phi^u\right)\end{bmatrix} \quad (30)$$

and

$$Q^{(1)}(\phi)=\det\left[B_2^{(1)}\phi^{2+s}-B_1^{(1)}\phi^{1+s}-M_2\left(\sum_{u=1}^{2}\left(D_u^{(1)}-E_u^{(1)}-L_u^{(1)}\right)\phi^u\right)\right] \quad (31)$$

$$Q^{(2)}(\phi)=\det\left[B_2^{(2)}\phi^{2+s}-B_1^{(2)}\phi^{1+s}-M_2\left(\sum_{u=1}^{2}\left(D_u^{(2)}-E_u^{(2)}-L_u^{(2)}\right)\phi^u\right)\right] \quad (32)$$

$$Q^{(3)}(\phi)=\det\left[B_2^{(3)}\phi^{2+s}-B_1^{(3)}\phi^{1+s}-M_2\left(\sum_{u=1}^{2}\left(D_u^{(3)}-E_u^{(3)}-L_u^{(3)}\right)\phi^u\right)\right] \quad (33)$$

Making use of Maple 17 and MATLAB, the region of Z - and W -stability for (14), (17) and (20) are shown in Fig.1 to 6.

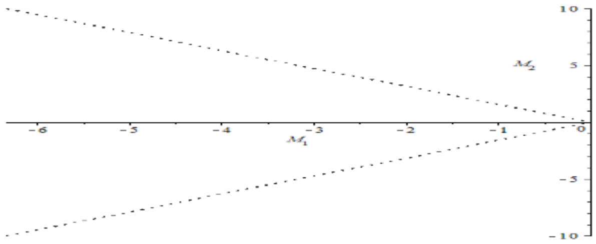

Fig.1.Region of Z -stability (ESDBBDFM) in (14)

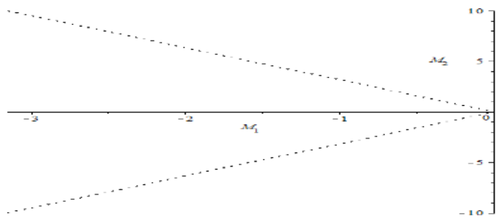

Fig.2.Region of Z -stability (ESDBBDFM) in (17)

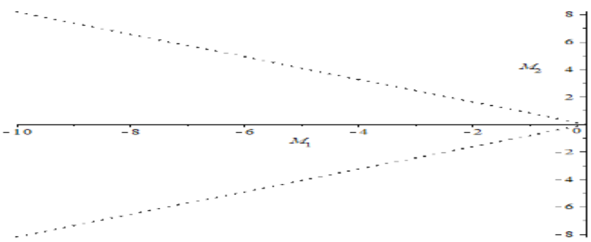

Fig.3.Region of Z -stability (ESDBBDFM) in (20)

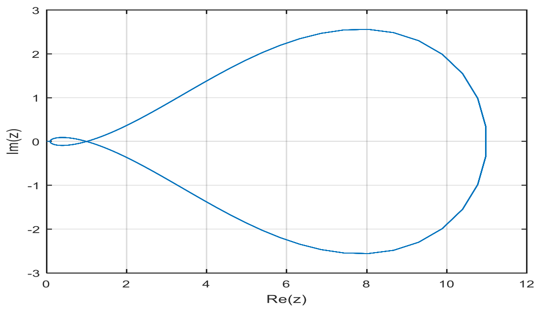

Fig.4. Region of W -stability (ESDBBDFM) in (14)

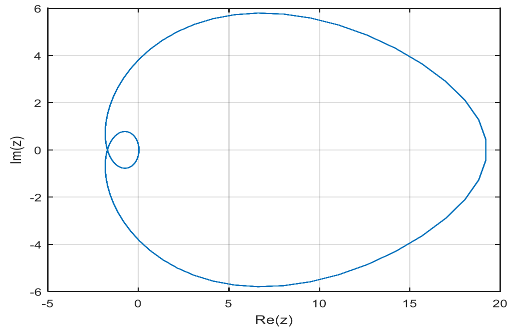

Fig.5. Region of W -stability (ESDBBDFM) in (17)

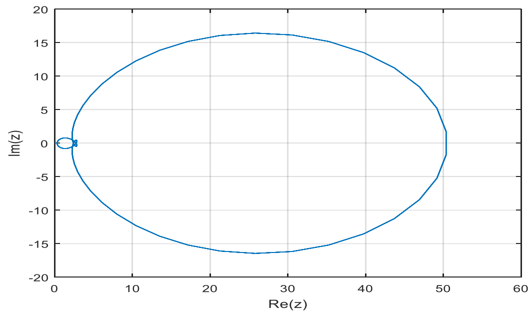

Fig.6. Region of W -stability (ESDBBDFM) in (20)

The Z -stability regions in Figs 1 to 3 lie inside the open-ended region while the W -stability regions in Figs 3 to 6 lie inside the enclosed region.

IV. NUMERICAL APPLICATIONS

In this section, some first-order delay differential equations shall be solved using (14), (17) and (20) of the discrete schemes been established. The delay term shall be calculated using the idea of sequence derived by [18].

J. Applications of ESDBBDFM to Solve Some First Order DDEs

Problem 1

$$x'(t)=-24x(t)-e^{(-25)}x(t-1),\ 0\le t\le 3$$

$$x(t)=e^{(-25)t},t\le 0$$

Exact solution $x(t)=e^{(-25)t}$

Problem 2

$$x'(t)=\cos(t)(x(x(t)-2)),\ 0\le t\le 3$$

$$x(t)=1,t\le 0$$

Exact Solution $x(t)=1+\sin(t)$

These problems were solved using the discrete schemes in (14), (17) and (20), and the solutions are presented below

TABLE 4.1.1: Approximate Solutions of Problem 1 for ESDBBDFM $k = 2,3$ and 4

T	Exact Solution	k = 2 Approximate Solution	k = 3 Approximate Solution	k = 4 Approximate Solution
0.01	0.778800783	0.738054333	0.845737424	0.707914084
0.02	0.60653066	0.581018457	0.653052175	0.555381082
0.03	0.472366553	0.430738175	0.513468959	0.430008728
0.04	0.367879441	0.317481402	0.378140722	0.337148612
0.05	0.286504797	0.249693367	0.319853832	0.245681157
0.06	0.22313016	0.184835774	0.247047534	0.17348597
0.07	0.173773943	0.13602625	0.194290129	0.135862998
0.08	0.135335283	0.106865461	0.143138782	0.104988665
0.09	0.105399225	0.078972253	0.121092862	0.082167968
0.1	0.082084999	0.058014885	0.093554596	0.059694048
0.11	0.063927861	0.045520285	0.073593603	0.04201957
0.12	0.049787068	0.033572236	0.05423944	0.032832814
0.13	0.038774208	0.0246119	0.045892315	0.025308953
0.14	0.030197383	0.019282717	0.035465404	0.019762073
0.15	0.023517746	0.014188339	0.027905203	0.014300864
0.16	0.018315639	0.01037612	0.020574521	0.010025531
0.17	0.014264234	0.008115167	0.017410785	0.007810652
0.18	0.011108997	0.00595467	0.013458671	0.006001304
0.19	0.008651695	0.004342028	0.010592249	0.004671808
0.2	0.006737947	0.003388769	0.007812727	0.003363272
0.21	0.005247518	0.002478281	0.006612342	0.002344934
0.22	0.004086771	0.001800716	0.005112793	0.001819649
0.23	0.003182781	0.001401779	0.004024854	0.001391977
0.24	0.002478752	0.001020951	0.002969852	0.001079108
0.25	0.001930454	0.000738572	0.00251392	0.000771306
0.26	0.001503439	0.000573109	0.001944346	0.000533662
0.27	0.00117088	0.000415263	0.001530986	0.000411796
0.28	0.000911882	0.00029874	0.001130123	0.000313031
0.29	0.000710174	0.000230866	0.000956767	0.000241217
0.3	0.000553084	0.000166171	0.000740197	0.000170608

TABLE 4.1.2: Approximate Solutions of Problem 2 for ESDBBDFM $k = 2,3$ and 4

t	Exact Solution	k = 2 Approximate Solution	k = 3 Approximate Solution	k = 4 Approximate Solution
0.01	1.009999833	1.010142329	1.009717661	1.010465815
0.02	1.019998667	1.020120782	1.019739789	1.020437051
0.03	1.0299955	1.030262832	1.02972058	1.030445892
0.04	1.039989334	1.040235898	1.039786271	1.040425843
0.05	1.049979169	1.050373612	1.049483505	1.050497676
0.06	1.059964006	1.060337301	1.059492497	1.06096945
0.07	1.069942847	1.070466624	1.069454703	1.070919453
0.08	1.079914694	1.080416952	1.079501049	1.080903846
0.09	1.089878549	1.090533832	1.089162343	1.090853196
0.1	1.099833417	1.100466818	1.099142187	1.100893732
0.11	1.109778301	1.110567209	1.109069872	1.111345291
0.12	1.119712207	1.120478879	1.119080799	1.121249191
0.13	1.129634143	1.130558741	1.128690697	1.131184182
0.14	1.139543115	1.140445132	1.138625427	1.14107806
0.15	1.149438132	1.150500433	1.148502708	1.151062205
0.16	1.159318207	1.16035759	1.158462201	1.161467425
0.17	1.169182349	1.170384308	1.168005329	1.171300467
0.18	1.179029573	1.180208288	1.177879052	1.181161223
0.19	1.188858895	1.190202413	1.187690127	1.190974901
0.2	1.198669331	1.199989287	1.197582254	1.2008777
0.21	1.2084599	1.209946821	1.207043345	1.211210575
0.22	1.218229623	1.219692675	1.216840265	1.220948182
0.23	1.227977524	1.229609635	1.226569439	1.230710055
0.24	1.237702626	1.239310569	1.236378374	1.240419004
0.25	1.247403959	1.249182988	1.245742292	1.250215705
0.26	1.257080552	1.258835124	1.255446737	1.260450407
0.27	1.266731437	1.268659054	1.265078444	1.27006824
0.28	1.276355649	1.278258529	1.274788495	1.279706831
0.29	1.285952225	1.28803004	1.284040259	1.289286785
0.3	1.295520207	1.297573015	1.293636702	1.2989529

V. RESULTS AND DISCUSSIONS

Here, the performances of the schemes derived in (14), (17) and (20), shall be applied in solving the two problems above by evaluating their absolute errors.

K. Analysis of Results

The analysis of results is obtained by computing absolute difference of exact solutions and numerical solutions. The results are summarized in the tables 5.1.1 to 5.1.2,

TABLE 5.1.1: Absolute Errors for ESDBBDFM $k=2,3$ and 4 using problem 1

t	k = 2 Error	k = 3 Error	k = 4 Error
0.01	0.04074645	0.066936641	0.070886699
0.02	0.025512203	0.046521515	0.051149578
0.03	0.041628378	0.041102406	0.042357825
0.04	0.050398039	0.010261281	0.030730829
0.05	0.03681143	0.033349035	0.04082364
0.06	0.038294386	0.023917374	0.04964419
0.07	0.037747693	0.020516186	0.037910946
0.08	0.028469822	0.007803499	0.030346618
0.09	0.026426971	0.015693637	0.023231256
0.1	0.024070114	0.011469597	0.02239095
0.11	0.018407576	0.009665742	0.021908292
0.12	0.016214833	0.004452371	0.016954254
0.13	0.014162308	0.007118107	0.013465255
0.14	0.010914667	0.005268021	0.010435311
0.15	0.009329407	0.004387457	0.009216882
0.16	0.007939519	0.002258883	0.008290108
0.17	0.006149067	0.003146551	0.006453582
0.18	0.005154326	0.002349674	0.005107692
0.19	0.004309667	0.001940554	0.003979887
0.2	0.003349178	0.00107478	0.003374675
0.21	0.002769238	0.001364823	0.002902585
0.22	0.002286056	0.001026022	0.002267122
0.23	0.001781002	0.000842073	0.001790803
0.24	0.001457801	0.000491099	0.001399644
0.25	0.001191882	0.000583465	0.001159149
0.26	0.00093033	0.000440906	0.000969778
0.27	0.000755617	0.000360106	0.000759084
0.28	0.000613142	0.000218241	0.000598851
0.29	0.000479308	0.000246593	0.000468958
0.3	0.000386914	0.000187112	0.000382476

TABLE 5.1.2: Absolute Errors of ESDBBDFM $k=2,3$ and 4 using problem 2

t	k = 2 Error	k = 3 Error	k = 4 Error
0.01	0.000142496	0.000282172	0.000465982
0.02	0.000122115	0.000258878	0.000438384
0.03	0.000267332	0.00027492	0.000450392
0.04	0.000246564	0.000203063	0.000436509
0.05	0.000394443	0.000495664	0.000518507
0.06	0.000373295	0.000471509	0.001005444
0.07	0.000523777	0.000488144	0.000976606
0.08	0.000502258	0.000413645	0.000989152
0.09	0.000655283	0.000716206	0.000974647
0.1	0.000633401	0.00069123	0.001060315
0.11	0.000788908	0.000708429	0.00156699
0.12	0.000766672	0.000631408	0.001536984
0.13	0.000924598	0.000943446	0.001550039
0.14	0.000902017	0.000917688	0.001534945
0.15	0.001062301	0.000935424	0.001624073
0.16	0.001039383	0.000856006	0.002149218
0.17	0.001201959	0.00117702	0.002118118
0.18	0.001178715	0.001150521	0.00213165
0.19	0.001343518	0.001168768	0.002116006
0.2	0.001319956	0.001087077	0.002208369
0.21	0.001486921	0.001416555	0.002750675
0.22	0.001463052	0.001389358	0.002718559
0.23	0.001632111	0.001408085	0.002732531
0.24	0.001607943	0.001324252	0.002716378
0.25	0.001779029	0.001661667	0.002811746
0.26	0.001754572	0.001633815	0.003369855
0.27	0.001927617	0.001652993	0.003336803
0.28	0.00190288	0.001567154	0.003351182
0.29	0.002077815	0.001911966	0.00333456
0.3	0.002052808	0.001883505	0.003432693

The notations used in the table below are stated as;

ESDBBDFM = Extended Second Derivative Block Backward Differentiation Formulae Methods for step numbers $k=2,3$ and 4.

SDBBDFM = Second Derivative Block Backward Differentiation Formulae Methods for step numbers $k=2,3$ and 4.

MAXE = Maximum Error.

TABLE 5.1.3: Comparison Between the Maximum Absolute Errors of ESDBBDFM and the SDBBDFM been studied by [19] on Numerical Treatment of first order DDEs using constant step size h = 0.01 for step numbers $k=2,3$ and 4 using Problem 1

Numerical Method	MAXE
ESDBBDFM MAXE for k = 2	5.04E-02
ESDBBDFM MAXE for k = 3	6.69E-02
ESDBBDFM MAXE for k = 4	7.09E-02
SDBBDFM MAXE for k = 2	5.42E-06
SDBBDFM MAXE for k = 3	2.43E-04
SDBBDFM MAXE for k = 4	0.001368661

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TABLE 5.1.4: Comparison Between the Maximum Absolute Errors of ESDBBDFM and the SDBBDFM been studied by [19] on Numerical Treatment of first order DDEs using constant step size h = 0.01 for step numbers $k = 2, 3$ and 4 using Problem 2

Numerical Method	MAXE
ESDBBDFM MAXE for k = 2	2.08E-03
ESDBBDFM MAXE for k = 3	1.91E-03
ESDBBDFM MAXE for k = 4	3.43E-03
SDBBDFM MAXE for k = 2	3.12E-05
SDBBDFM MAXE for k = 3	9.63E-04
SDBBDFM MAXE for k = 4	0.001824049

Using Microsoft Excel, the MAXE for ESDBBDFM and SDBBDFM for Problem 1 and 2 are presented as

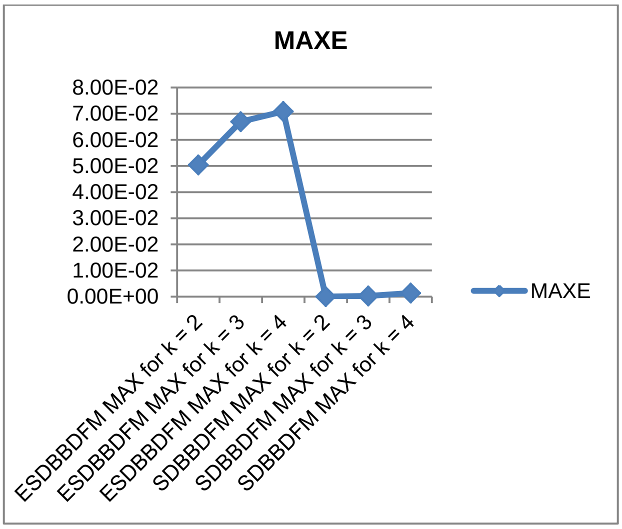

Fig.7. MAXE for ESDBBDFM and SDBBDFM (Osu et al, 2020) for Problem 1

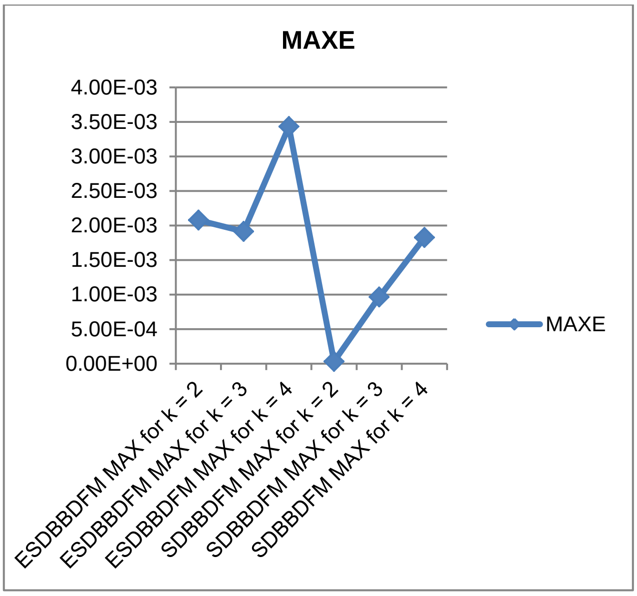

Fig.8. MAXE for ESDBBDFM and SDBBDFM (Osu et al, 2020) for Problem 2

VI. CONCLUSIONS

In conclusion, the discrete schemes of (14), (17) and (20) were worked-out from their individual continuous expressions and were found to be more efficient, accurate, consistent, convergent, Z-stable and W-stable when compared with other existing method. Also, it was revealed in tables 5.1.1 to 5.1.2 that the ESDBBDFM for $k = 4$ scheme performed better than the ESDBBDFM schemes of step numbers $k = 3$ and $k = 2$ respectively. Therefore, it is recommended that ESDBBDFM schemes for step numbers $k = 2, 3,$ and 4 are suitable for solving DDEs. It is also recommended that the ESDBBDFM schemes of higher step numbers perform better than the ESDBBDFM schemes of lower step numbers. Further study should be carried out for step number $k = 5, 6, 7...$ on the construction of discrete schemes of ESDBBDFM for solving DDEs without the use of interpolation techniques in evaluating their delay terms.

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