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A Possible Solution to a Relativistic Orbit for the 2-Body Celestial Mechanics Problem

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***Abstract*— The 'captured' 2-body Kepler problem considering relativity is solved by using an iterative integral equation. The rationale for this approach is to increase the accuracy of the limits for a satellite's motion and potentially provide a test to evaluate different gravitational laws. Moreover, this effort might provide additional insights to resolve other anomalies such as the flyby anomaly, the Faint Young Sun paradox, the Pioneer anomaly and other inconsistencies that potentially may be used to validate Einstein's Theory of Relativity. The mathematical solution solves a nonlinear Volterra integral equation using an iterative fashion, which reveals a correction factor for treating a given closed orbit. However. This correction factor is not a constant value but rather a function of the elliptical or circular orbit angular displacement. This function may be insignificant during portions of the trajectory, say at apogees or perigees. Nonetheless, these results are encouraging where relativity effects may or may not exist to understand and resolve several of the other variances or gravitational anomalies currently related to our solar system.**

I. INTRODUCTION

There is a need to develop a testing function to examine gravitational anomalies. The question of interest for one situation will examine: if relativity has a major impact on a spacecraft's orbit? This trajectory involves a nonlinear Volterra integral equation. In the future, these insights with these other compared. Iorio, for example, provides an excellent examination of these anomalies that are worth examining. Moreover, relativity effects may somehow also influence some of these anomalies as well.

To paraphrase Iorio and include some of these views, there are currently accepted laws of gravitation applied to known bodies which may have the potential of paving the way for remarkable advances in fundamental physics. This is particularly important now more than ever, given that most of the Universe seem to be made of unknown substances labeled Dark Matter and Dark Energy or, by contrast, one may solve these issues by using a different gravitational law other than Newtonian gravitation. Moreover, investigations in one of such directions can seek destiny to enrich and find other solutions as well. The current status of some of these alleged gravitational anomalies in the Solar system are:

a) Possible anomalous advances of *planetary perihelia*,
b) Unexplained orbital residuals of a recently *discovered moon of Uranus* (Mab),
c) The lingering unexplained secular increase of the *eccentricity of the orbit of the Moon*,
d) The so-called *Faint Young Sun Paradox*,
e) The secular decrease of the mass parameter of the Sun,
f) The *Flyby Anomaly*,
g) The *Pioneer Anomaly*, and
h) The anomalous secular increase of the astronomical unit.

One more anomaly should also be added to include the Trojan asteroids placed in Libration anomalies, say from Iorio [1], should be points from the Sun-Jupiter system discussed by the author in [2] which may offer an opportunity for demonstrating gravitational waves as well as gravitational repulsion.

Basically, anomalies [1] may show up in experiments to make comparisons with the conventional wisdom. In science, the word 'anomaly' designates some sort of discrepancies with respect to an expected path observed in systematic errors in the observations. Once determined that the anomaly is real, honest engineers and scientists need to determine given phenomenon. In astronomical contexts, it was used since ancient times to indicate irregularities in motions of celestial objects. First of all, it must be carefully ascertained if the anomaly really exists; it may be either a mere artifact of the data reduction procedure, or the consequence of malfunctioning of the measuring devices and how to change the conventional wisdom for explaining the anomaly as well as still accept and preserve the existing body of knowledge. Let us address some of these anomalies [1] with the objective that there may be a serious need to change the kinematic motions to explain some of these events.

Anomalous Secular Increase of the ***Eccentricity of the Moon's Orbit*** demonstrated steady progress in reviewing data from the Lunar Laser Ranging (LLR) technique. In the last decades, data has determined the orbital changes at a cm level of accuracy or better which allows for accurate testing of the General Theory of Relativity. Moreover, a major limiting factor in our knowledge of the celestial course of the Moon is currently based by a description of the complex geophysical processes. The lingering unexplained increase of the eccentricity of the Moon's orbit is yet to be understood, despite recent efforts to improve the geophysical models of the intricate tidal phenomena taking place in the interior of our planet and its natural satellite.

The eccentricity rate $\dot{e}$ can vary from a high of $1.6 \pm 0.5 \times 10^{-11}$ yr^{-1} to a low of $9 \pm 3 \times 10^{-12}$ yr^{-1}. The general relativistic Lense-Thirring acceleration induced by the Earth's gravitomagnetic field acting on the Moon has the correct order of magnitude, but it does not affect *e*. A still undetected distant planet in the Solar system does, in principle, make *e* cumulatively change over time, but the required mass and distance for it are yet to be determined.

Anomalous perihelion precession of Mercury of 42.98 "cy^{-1} (a change of its orbit by 42.98 arc seconds in a century) since it is nowadays fully included in the state-of-the-art models of all of the modern ephemerides. Instead, if real, it would be due to some unmodeled dynamical effects [3]-[4] which, in principle, could potentially signal a breakthrough with the currently accepted laws of gravitation. The relativistic dynamical models for the modern ephemerides, for example, Mercury, are not complete that do not include the 1PN gravitomagnetic field of the Sun, not to say of the other major bodies of the Solar system, which causes the Lense-Thirring effect [5]. This effect would be comparable to the action of a hypothetical ring of undetected moonlets in its neighborhood as a possible solution using conventional gravitational physics regarding the gravitational anomalies for Uranus.

The ***Faint Young Sun Paradox***: According to established evolutionary models [6]-[8] of the Sun's history, the energy output of our star during the Archean, from 3.8 to 2.5 Gyr ago, would have been insufficient to maintain liquid water on the Earth's surface. Instead, there are strong but compelling independent evidence where our planet was mainly covered by liquid water oceans, hosting also forms of life, during that remote era. As such, our planet could not be entirely frozen during such an era, where it would have necessarily been if it received only about 75% of the current solar irradiance. One view implies a steady precession of the Earth's orbit during the entire Archean eon provided a closer location to its present day heliocentric distance in such a way when the Sun's luminosity was adequate. Thus the effects of the ocean may be a potential gravitational anomaly.

As ***'Flyby Anomaly'*** [9]-[13] is intended to treat the collection of unexplained increases for v_∞ in the asymptotic line-of-sight velocity in the direction of v_∞. This has been of the order of ≈ 1 − 10 mm s^{-1} with uncertainties to as little as ≈ 0.05 − 0.1 mm s^{-1}, which have been experienced by the interplanetary spacecraft Galileo, NEAR, Cassini, Rosetta and, perhaps, Juno during their Earth flybys. The flyby anomalies have not yet been detected when such spacecraft flew by other planets. This perhaps may be due to their still relatively inaccurate gravity field models compared to the Earth's gravitational model.

Pioneer Anomaly*:* At the end of the twentieth century, it was reported that radio tracking data from the Pioneer 10 and 11 spacecraft [14]-[16] exhibited a small anomalous blue-shifted frequency drift uniformly changed the rate of $5.99 \pm 0.01 \times 10^{-9}$ Hz s^{-1} interpreted as a constant and uniform deceleration approximately directed towards the Sun. This was at heliocentric distances approximately of 20 − 70 au. Each satellite moved in opposite direction from the sun to determine if other effects such as the solar wind would exist.

Subsequent years witnessed some options to explain a variety of conventional and exotic physical mechanisms for both gravitational and non-gravitational nature for these differences. These gravitational effects started when the instrumentation was stopped and the electrical power from a nuclear isotope power supply was altered in a different electrical circuit involving a heater to dissipate electrical energy. In 2012, an appropriate model of the recoil force assumed that an anisotropic emission of thermal radiation off the spacecraft was able to accommodate for about 80% of the unexplained acceleration plaguing the telemetry of both the Pioneer probes as far as magnitude, temporal behavior, and direction of concern. The remaining 20% still does not represent a statistically significant anomaly in view of uncertainties in the acceleration estimates using Doppler telemetry and thermal models. On the other hand, the Pioneer anomaly may be due to some exotic gravitational mechanism external to the spacecraft. This resulted in the form of a constant value and uniform acceleration directed towards the Sun. These views were performed with systematic investigations about its presumed effects on bodies other than the Pioneer probes performed since 2006.

It turned out the Pioneer anomaly may also involve induced anomalous signatures of Uranus, Neptune, and Pluto. This would be far too large to consider the initial conditions or strong tensions between the Galactic tide dominant in making Oort cloud comets observable. The action may be a putative Pioneer anomaly-like acceleration in those remote peripheries of the Solar system.

Thus, these anomalies [1] regarding the standard behavior of natural and artificial systems within the Sun's realm as expected may consider where the conventional physics possesses a great potential to uncover modifications of our currently accepted picture of natural laws. Nonetheless, before this dream really comes true, it is mandatory that the unexpected patterns are confirmed to an adequate level of statistical significance by independent analyses, and any possible conventional viable mechanism could be responsible can be reliably excluded.

With these thoughts regarding anomalies, the equations of motion for the two-body celestial mechanics problem, although well-established, is altered. Additionally, several references [17]-[18] have provided supplementary factors to include changes in the trajectories with the Theory of Relativity. One wonders about the magnitude and the impact to these trajectories if for a probe of interest, moving at conditions for an elliptical orbit, which would exert some changes due to relativity. Moreover, the concern should also focus on trajectory changes if the probe is moving at or near the speed of light. However, we cannot deal with this latter problem at this time. Under these circumstances, light speed trajectories may alter gravitational forces themselves as well as treating only with relativity.

II. DISCUSSION

Jefimenko [17]-[18] looked at producing a gravitational law that allows for a probe moving at the speed of light considering relativity specifically due to gravity. As expected, the model result at slower speeds is asymptotic with Newtonian gravity typical to a probe in orbit around the Earth or moving in the near-abroad within our solar system. What may be of concern is the inherent trajectory may still have some sensitivity to relativity while the gravitational law may be a bother based upon affects near the speed of light. Thus this evaluation may provide some insights toward using a testing ground to evaluate different gravitational laws comparable to Newtonian gravitation.

In previous studies by the author [19], a Green's function solution was treated for the trajectories in a binary pulsar. The interesting factor within the discipline of astrophysics, states these systems are usually identified with a single eccentricity value for both celestial bodies. These results [19] indicate for most of these binary systems, the trajectories may not be duplicates of each other for a neutron star and its companion. There are some situations, which could result where one body is in a nearly circular orbit while the other companion or neutron star is clearly in an elliptical orbit. Such situations for these orbital trajectories demand separate eccentricities as demonstrated by Figure 1.

Results of this assessment suggest the different eccentricities for binary pulsars with a neutron star and a companion should be:

$$e_1 = \left(\frac{2h_1^2}{\mu_1^3 r_1^o} - 1\right)\Big/\cos\theta_o, \quad and$$

$$e_2 = \left(\frac{2h_2^2}{\mu_2^3 r_2^o} - 1\right)\Big/\cos\left(\theta_o + \pi\right) = \left(\frac{2h_1^2 \mu_2}{\mu_1^4 r_2^o} - 1\right)\Big/\cos\left(\theta_o + \pi\right).$$

These eccentricity differences (***e**$_1$* and ***e**$_2$*) are a function of several parameters such as the angular momentum per unit mass (***h***), initial orbital parameters (**θ_o**) as well as weight or gravitation.

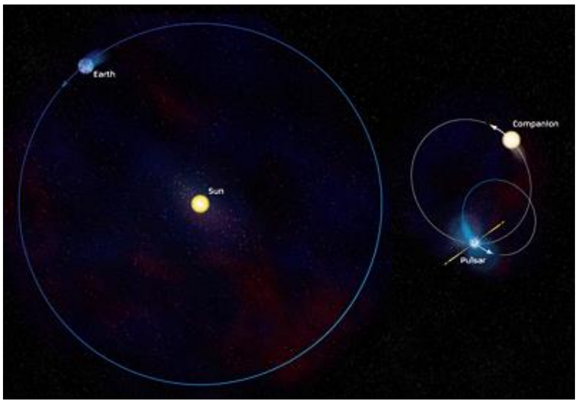

FIGURE 1. The trajectories for the pulsar J1903+0327 with its possible sun-like companion star compared with the orbit of the Earth around the sun.

III. ANALYSIS

Before looking at specific orbits and orientations, several notions are required. This will be discussed for a basic understanding shown with typical orbit definition, a Green's function, the impact of relativity to the trajectory, and finally a solution to the integral equation for relativity trajectory.

A. Standard Terminology

The basic problem of the 'captured' two-body model is where one body is relatively light in terms of mass while the other body has a significant mass. Motion is in the same plane simplify the mathematics of the problem. With this premise [20]-[21], the body with the larger mass is assumed to be immovable compared to the first body. The issue is to determine the initial momentum conditions and energy conservation problem essentially based on the premise where the lighter body performs the dynamics of interest while the larger body is assumed stationary. There are a distance and an angular orientation to completely specify the coordinate location related to the reference coordinate origin. The distance between the two bodies is from a center of the reference coordinate system includes a focal point on the smaller mass' orbit. The center of this reference coordinate system is assumed to also be some negligible distance from the center of the larger body. In reality, there is some small distance treated as inconsequential between the actual weight locations for the barycenter.

The radial and angular momentum equation for the smaller body is defined as:

$$\bar{F}_r = m\bar{a}_r = m\left(\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\hat{r},$$
$$\bar{F}_\theta = m\bar{a}_\theta = m\left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\hat{\theta} = \frac{m}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\hat{\theta}. \quad (1)$$

Distance from the Barycenter (center of mass for these bodies) is measured to the spacecraft with ***r***. Carat symbols represent the unit direction normal to the trajectory with r and tangentially with ***θ***. Time is measured with ***t***. The subscripts in the LHS are not derivatives but the radial and azimuthal force directions respectively in the coordinate system. Derivatives are functions of time. The radial force includes the gravitational attraction between the two bodies. Moreover, the second equation assumes where the azimuthal force vanishes for each of these bodies.

In this problem, the radial dimension is changed as the difference between the distances to the two objects. The problem can be reduced to one dimension with some definitions where ***μ*** is the total mass of both bodies (***G*** ***(m_1+m_2)***), ***G*** is the universal gravitational constant, and ***m*** is mass. This is considered as the gravitational attraction for this problem as follows:

$$\left(\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right) = -\frac{\mu}{r^2},$$
$$m\left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right) = \frac{m}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0., \quad h = r^2\left(\frac{d\theta}{dt}\right). \quad (2)$$

Clearly, the azimuthal gravitation disappears with a constant, ***h***, that is the angular momentum per unit mass to satisfy the azimuthal acceleration. Thus the second equation vanishes. A variable ***u*** is selected based upon an inverse function of the radius to simplify the problem and removing the time derivatives to account for azimuthal derivatives with substitutions from the problem. This results in:

$$r = \frac{1}{u}, \quad \frac{d\theta}{dt} = hu^2, \quad \frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -h\frac{du}{d\theta}; \quad \frac{d^2 r}{dt^2} = -h\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -h^2u^2\frac{d^2u}{d\theta^2}. \quad (3)$$

When these are substituted into the above equation for the radial momentum, the results are:

$$-h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3 = -\mu u^2 \ , \quad (4)$$

or with some simplifications:

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} \quad (5)$$

This is a key equation for this problem. The solution of this ordinary differential equation considering a geometric length ***l*** and eccentricity ***e*** is:

$$u = \frac{\mu}{h^2} + C\cos(\theta - \theta_o) \quad or \quad r = \frac{l}{(1 + e\cos\theta)}. \quad (6)$$

The importance of this equation is that the eccentricity ***e*** plays a significant role. Basically, the smaller body rotates about the larger body with a circular orbit (if ***e*** is zero) or if the eccentricity is positive and less than *1.0*, the orbit is an elliptical orbit with the major body located at one of the focal points in the elliptical orbit. If the eccentricity is greater than *1.0*, the orbit is hyperbolic and it leaves or escapes the gravitational pull of the larger body. Obviously, this result depends upon initial velocity conditions and kinetic energy before the interaction.

A solution to this integral equation using a Green's function solution [22] accounting for boundary conditions is:

$$u(\theta) = C_1\cos(\theta - \theta_o) + \frac{\mu}{h^2}\left\{\int_{\theta_o}^{\theta}\sin(\theta - \xi)\,d\xi + \frac{\cos\theta}{2\sin\pi/2}\int_{\theta_o}^{\theta_o+2\pi}\cos\left(\xi - \pi/2\right)d\xi + \frac{\sin\theta}{2\sin\pi/2}\int_{\theta_o}^{\theta_o+2\pi}\sin\left(\xi - \pi/2\right)\,d\xi\right\}. \quad (7)$$

These terms are the solution for the above equation where the Green's function will become the kernel of the integral equation which is defined as:

$$u(\theta) = C_1 \cos(\theta-\theta_o) + \alpha K(\theta,\theta), \quad (8)$$

$$\text{where: } K(\theta,\theta) = \left\{ \int_{\theta_o}^{\theta} \sin(\theta-\xi)\, d\xi + \frac{\cos\theta}{2\sin\pi/2} \int_{\theta_o}^{\theta_o+2\pi} \cos\left(\xi - \pi/2\right) d\xi + \frac{\sin\theta}{2\sin\pi/2} \int_{\theta_o}^{\theta_o+2\pi} \sin\left(\xi-\pi/2\right)\, d\xi \right\}$$

This is the basic solution to the problem without relativistic effects where the solution is the same as equation 6.

B. Relativistic Mechanics

Relativistic effects can vary the sense of time dilation and changes in length. Such changes depend upon the velocity. Let our probe move at a stationary orbit about the Earth. The probe's trajectory can be given for a geodesic [23] in:

$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma} \frac{d x^\beta}{d\tau} \frac{d x^\gamma}{d\tau} = 0 . \quad (9)$$

where τ is the proper time and x is a linear measure. This is rewritten as:

$$\frac{d}{d\tau}\left(g_{\lambda\nu} \frac{d x^\nu}{d\tau}\right) - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\lambda} \frac{d x^\mu}{d\tau} \frac{d x^\nu}{d\tau} = 0 . \quad (10)$$

The proper time is based upon the space-time interval that depends upon the metric:

$$ds^2 = -c^2 d\tau^2 = g_{\mu\nu} d x^\mu d x^\nu . \quad (11)$$

Considering the Sun's gravitational potential having a mass of M, the Schwartzschild metric using standard coordinates in a spherical coordinate system is:

$$ds^2 = -c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right)^{-1} d r^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) - \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 . \quad (12)$$

(12)

After considerable terms and assumptions related to a plane based upon Newtonian Mechanics and defining constants of integration, this becomes:

$$\left(\frac{h}{r^2}\frac{dr}{d\theta}\right)^2 = c^2 (k-1) + \frac{2GM}{r} - \frac{h^2}{r^2}\left(1 - \frac{2GM}{c^2 r}\right) . \quad (13)$$

If you allow $u = 1/r$ as previously and obtain $dr/d\theta = -r^2\, du/d\theta$ as in the original definition, this results into:

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = c^2 \frac{(k-1)}{h^2} + \frac{2GM}{h^2} u + \frac{2GM}{c^2} u^3 \quad . \quad (14)$$

Differentiating this equation results in:

$$\frac{d^2 u}{d\theta^2} + u = \frac{GM}{h^2} + 3\frac{GM}{c^2} u^2 = \alpha - \beta u^2 . \quad (15)$$

Note the value of β is basically from the Theory of Relativity and is small which is why this effect is not usually considered regarding short-term celestial mechanics. However, do we fully understand the impact of this value with respect to a given trajectory? Let us consider this point.

C. Solution Rationale

Using relativity, the integral equation for the above ordinary differential equation is of interest as:

$$u(\theta) = C_1 \cos(\theta-\theta_o) + \alpha K(\theta,\theta) - \beta \int_0^\vartheta K(\theta,\xi) u^2(\xi)\, d\xi = \varsigma(\theta,\theta) - \beta \int_0^\vartheta K(\theta,\xi) u^2(\xi)\, d\xi . \quad (16)$$

This is an inhomogeneous Fredholm equation or a Volterra integral equation. Because of the squared term[1] for the independent variable and the coupling between these terms, this is nonlinear. The first term is a previously determined orbit trajectory solution without relativity. Here, we are assuming this tends to minimize the coupling impact with the integral equation.

Normally in using an iterative process, an initial equation (the first term) is assumed as a starting position. This is then used in the next correction using the recent previous values and the process continues until you can identify specific terms that represent solutions. The process is used differently here where the initially assumed value is the orbital trajectory solution without relativity. In this fashion, this is similar to evaluating a perturbation to the results using the nonlinear effects for corrections. The first term on the RHS represents using an iterative process to start the solution. This process is performed in a similar iterative fashion; however, some terms will not be included because of orders of magnitude effects. This means for generating a series if it exists, the absolute magnitude of the kernel is basically less than the value of 1 and with numerous K^n terms, this becomes insignificant.

[1] It is interesting to note where the form of this equation may also solve a problem in fluid dynamics as well as with other disciplines. Solving this problem, therefore is of value. Furthermore, the approach of using integral equations offers additional tools resolving mathematical physic challenges.

This would be used to asymptotically approach the answer. The process for ***u*** starts with ***u₀*** with the first iterative and so on as ***u₁*** on up to ***u₂*** and ***u₃***:

$$u_0(\theta) = \varsigma - \beta \int_0^{\vartheta} K(\theta,\xi) u_0^{\,2}(\xi)\, d\xi, \quad or$$

$$u_1(\theta) = \varsigma - \beta \int_0^{\vartheta} K(\theta,\xi)\left[\varsigma - \beta \int_0^{\vartheta} K(\theta,\xi) u_0^{\,2}(\xi)\, d\xi.\right]^2 d\xi. \tag{17}$$

$$then: \; u_1(\theta) = \zeta - \beta\, K(\theta,\theta)\, \zeta^2 + 2\zeta \beta^2 K \int_0^{\vartheta} K(\theta,\xi) u_0^{\,2}(\xi)\, d\xi + \ldots \tag{18}$$

This is essentially the solution without relativity. Using this as another iteration, this becomes:

$$u_2(\theta) = \zeta - \beta\, K(\theta,\theta)\, \zeta^2 + \zeta \beta^2 K \int_0^{\vartheta} K(\theta,\xi)\left[\zeta - \beta\, K(\theta,\theta)\, \zeta^2 + 2\zeta \beta^2 K \int_0^{\vartheta} K(\theta,\xi) u_1^{\,2}(\xi)\, d\xi + \ldots\right]^2 d\xi + \ldots \quad (19$$

Let: $\varphi = \beta K \varsigma$, then the next iteration results in:

$$u_3(\theta) = \frac{1}{\beta K}\left[\, \varphi - \varphi^2 + 2\varphi^3 - 4\varphi^4 + 2\varphi^5 + 8\frac{\varphi^3}{K^2}\left(\varphi - \varphi^2\right)\right]\int_0^{\vartheta} K(\theta,\xi) u_2^{\,2}(\xi)\, d\xi + \ldots \tag{20}$$

This becomes:

$$u_3(\theta) = \frac{\varphi}{\beta\; K}\left[\, 1 - \varphi \left[\, 1 - 2\varphi + 2^2\varphi^2 - 2^3\varphi^3 + 2^4\varphi^4 - 2^5\varphi^5 + \;\ldots\ldots \right.\right. \tag{21}$$

Or for the ***n****th* iteration, this is a series where:

$$u_n(\theta) = \frac{\varphi}{\beta K}\left[\, 1 - \varphi \left[\, 1 - 2\varphi + 2^2\varphi^2 - 2^3\varphi^3 + 2^4\varphi^4 - 2^5\varphi^5 + \ldots\ldots\ldots \pm 2^{n-1}\varphi^{n-1} \right.\right. . \tag{22}$$

Simplifying this results[2] in:

$$u_n(\theta) = \frac{\varphi}{\beta K}\left[1 - \frac{\varphi}{1+2\varphi}\right] = \varsigma \left[1 - \frac{\varphi}{1+2\varphi}\right]. \tag{23}$$

This is the final equation or trajectory solution with relativistic effects. Note that the term with unity is the trajectory without relativistic effects and the other term is considered as a correction. This can be further simplified in a form to separate the elliptical trajectory results with the objective of creating a correction factor:

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[2] There are some interesting points for consideration for this mathematical solution. For example, the basic solution to the linear integral equation looks like:

$u(x) = f(x) + \lambda \int K(x,s)u(s)ds \; is \; u(x) = \frac{\lambda \int K(x,s)f(s)ds}{1-\lambda \int K(x,s)f(s)ds}$. The solution for this problem is found:

$$u(x) = f(x) + \lambda \int K(x,s)u^2(s)ds \; is \; u(x) = \frac{\lambda \int K(x,s)f(s)ds}{1 + 2\lambda \int K(x,s)f(s)ds}.$$

This can be extended most likely as follows:

$$u(x) = f(x) + \lambda \int K(x,s)u^n(s)ds \; is \; u(x) = \frac{\lambda \int K(x,s)f(s)ds}{1 + (-1)^n n\lambda \int K(x,s)f(s)ds}.$$

$$u(\theta)=\varsigma\left[\frac{(1+K(\theta,\theta))+\beta\alpha K(\theta,\theta)}{(1+K(\theta,\theta))+2\beta\alpha K(\theta,\theta)}\right]=\varsigma\left[\frac{1+K(\theta,\theta)(1+\beta\alpha)}{1+K(\theta,\theta)(1+2\beta\alpha)}\right] \quad (24)$$

Note the first term in the RHS is the elliptical trajectory equation without relativity effects. The right-hand term on the RHS represents a correction factor to allow for relativity. The term is general enough to include a Green's function using an initial value or a boundary value problem.

This results in:

$$u(\theta)=\varsigma\left[\frac{1+K(\theta,\theta)\left(1-3\left(\frac{GM}{hc}\right)^2\right)}{1+K(\theta,\theta)\left(1-6\left(\frac{GM}{hc}\right)^2\right)}\right]=\varsigma\,\psi \quad (25)$$

Thus this includes the original trajectory with a correction factor. The variable ψ is the desired correction factor that, because of the kernel, will not be a constant function but rather a function of angular displacement during the orbit. For this, the integral equation kernel can be used for initial value or boundary condition problems. The kernel in the above problem where the initial angle θ_0 is zero will become:

$$K(\theta,\theta)=\left\{\int_{\theta_0}^{\theta}\sin(\theta-\xi)\,d\xi+\frac{\cos\theta}{2\sin\pi/2}\int_{\theta_0}^{\theta_0+2\pi}\cos\left(\xi-\pi/2\right)d\xi+\frac{\sin\theta}{2\sin\pi/2}\int_{\theta_0}^{\theta_0+2\pi}\sin\left(\xi-\pi/2\right)d\xi\right\} \quad (26)$$

or for these conditions:

$$K(\theta,\theta)=1-\cos\theta. \quad (27)$$

And thus the correction factor is:

$$\psi(\theta)=\left[\frac{1+(1-\cos\theta)\left(1-3\left(\frac{GM}{hc}\right)^2\right)}{1+(1-\cos\theta)\left(1-6\left(\frac{GM}{hc}\right)^2\right)}\right]. \quad (28)$$

The problem is when the numbers are included for a circular stationary orbit, the multipliers involve 11 decimal places of 9 such as .9999999… onwards before different numerical values appear. This is the right-hand terms on the numerator and denominator. Clearly, since we do not have the sensitivity for these values, ψ can easily be assumed to be unity and for all practical purposes, the solution for the two body problem is more than adequate in terms of accuracy to ignore relativity. On this basis, it is clear the problem of Mercury's perihelion required a measure of small minute amounts of azimuthal change over a considerable century to assess the contributions from the theory of relativity. Moreover, additional research may provide some rationales to explain the other anomalies previously mentioned.

IV. CONCLUSIONS

The purpose of this effort was to determine a potential means for predicting a testing function to assess different gravitational laws hopefully with examining a closed-loop trajectory. After considerable mathematics to treat the complexity of this problem nonlinear Volterra integral equation, a factor was identified that would provide the trajectory without relativity compared to the same trajectory with relativity. The resulting factor appeared to be a function of the azimuthal direction which occurs during an elliptical orbit; however, this azimuthal effect may be of a small consequence unless there is adequate sensitivity for these effects. This result opens the door for explaining several of the anomalous behavior of the solar system.

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