Convergence Of Discrete Scheme For The Quenching Solution Of A Semilinear Parabolic PDE With A Singular Nonlinearity

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Abstract—This paper concerns the study of the numerical approximations for the following boundary value problem:

\[
\begin{cases}
    u_t(x,t) + u_{xx}(x,t) - c(x)f(u(x,t)) & x \in (0,1), t \in (0,T), \\
    u_x(0,t) = 0, & t \in (0,T), \\
    u(0) = u_0(x) > 0, & x \in [0,1]
\end{cases}
\]

Where \( f : (0, \infty) \to (0, \infty) \) is \( C^1 \) convex nondecreasing function, \( \lim_{s \to 0^+} f(s) = +\infty, \int_0^s \frac{ds}{f(s)} < \infty \) for any positive real \( \alpha \). The initial datum \( u_0 \in C^2([0,1]), u_0(0) = 0 \) and \( u_0'(1) = 0 \). The potential \( c \in C^1([0,1]), c(x) > 0, x \in (0,1), c'(0) = 0, c'(1) = 0 \).

We find some conditions under which the solution of a semidiscrete form of the above problem quenches a finite time and estimate its semidiscrete quenching time. We also the semidiscrete quenching time converges to the real one when the mesh size goes to zero. A similar study has been also investigated a discrete form of the above problem. Finally, we give some numerical experiments to illustrate our analysis.


Key-words and phrase — Semidiscretizations, Semilinear heat Equation, quenching, numerical quenching time, convergence, Full discretizations.

1. Introduction

Consider the following boundary value problem

\[
\begin{cases}
    u_t(x,t) + u_{xx}(x,t) - c(x)f(u(x,t)) & x \in (0,1), t \in (0,T) \\
    u_x(0,t) = 0, & t \in (0,T) \\
    u(0) = u_0(x) > 0, & x \in [0,1]
\end{cases}
\]

Where \( f : (0, \infty) \to (0, \infty) \) is \( C^1 \) non decreasing function, and \( C^1 \) convexe function, \( \int_0^a \frac{ds}{f(s)} < \infty \) for any positive real \( \alpha \). \( \lim_{s \to 0^+} f(s) = +\infty, \)

\( c \in C^1([0,1]), c(x) > 0, x \in (0,1), c'(0) = 0, c'(1) = 0, u_0(0) = 0, u_0'(1) = 0 \),

and \( u'(1) = 0 \), the initial data \( u_0 \in C^2([0,1]) \),

\( u_0(x) > 0, x \in (0,1), u_0'(0) = 0, u_0'(1) = 0. \)

Here \([0,T]\) is the maximal time interval on which the solution \( u \) of (1)-(3) exist. The time \( T \) may be finite or infinite. When \( T \) is finite, then we say that the
solution of (1)-(3) develops a singularity in a finite time, namely, \( \lim_{t \to T^-} U_{\text{min}}(t) = 0 \).

Where \( U_{\text{min}}(t) = \min_{0 \leq s \leq t} U(x,t) \).

In this last case, we say that the solution of (1)-(3) quenches in a finite time and the time \( T \) is called the quenching of the solution \( u \).

The theoretical study of solution for semi linear heat equations which quench in finite time has been the subject of investigation of many authors see [2], [7], [11], [13], [21], [22], [18], [30], and the reference of classical solution has been proved and this solution is unique. In addition, it is shown that if the initial data at (3) satisfies

\[
u_0(x) - c(x)u_0^q(x) \leq -Bu_0^q(x), \quad x \in [0,1]
\]

Where \( B \in (0,1) \) and \( q > 0 \), then the classical solution of (1)-(3) Quenches in finite time \( T \) and we have the following estimate

\[
\min_{0 \leq s \leq t} (u_0(x))^q \leq \frac{1}{q+1} \leq L \leq \frac{\min_{0 \leq s \leq t} (u_0(x))^q}{B(q+1)},
\]

\[
\frac{1}{(q+1)^{q+1}} \leq \min_{0 \leq s \leq t} (t) \leq \frac{1}{q+1} \frac{1}{(q+1)^{q+1}} \leq \min_{0 \leq s \leq t} (T-t) \leq \frac{1}{q+1} \frac{1}{(q+1)^{q+1}}
\]

(See, For instance ([7], [11], [12])).

In this article, we are interested in the numerical study of the phenomenon of quenching. Our aim is to build a semidiscrete scheme where solution obeys the property of the continuous one. In order to facilitate our discussion, let us notice that the first condition in (3) allows the solution \( u \) to attain its minimum at the point \( x = 0 \), and the second one permits the solution \( u \) to decrease with respect to the second variable.

The hypotheses (5) are compatibility condition which ensure the regularity of the solution. The hypotheses (6) are compatibility condition which ensure the regularity of the solution.

This paper is organized as follows. In the next section, we give some results about the discrete maximum principle. In the third section, under some conditions, we prove that the solution of a semidiscrete form of (1) --(3) quenches in a finite time and estimate its semidiscrete quenching time. In the fourth section, we prove the convergence of the semidiscrete quenching time. In the fifth section, we study the results of sections 3 and 4 taking a discrete form of (1) --(3). Finally, in the last section, we give some numerical results to illustrate our analysis.

2- Properties of semi discrete problem

We start our study by the construction of a semidiscrete scheme as follows. Let \( I \) be a positive integer, and let \( h = \frac{1}{I} \). Define the grid

\[ x_i = ih, 0 \leq i \leq I, \]

and approximate the solution \( U_i(t) = (U_0(t),U_1(t), \ldots, U_I(t))^T \) of the following semidiscrete equations

\[
\frac{dU_i(t)}{dt} - \delta^2 U_i(t) = -\beta_i f(U_i(t)), 0 \leq i \leq I, t\in(0, T_q^h),
\]

(7)

\[
U_i(t) = 1, t\in(0, T_q^h),
\]

(8)

\[
U_i(t) = \varphi_i > 0, 0 \leq i \leq I,
\]

(9)

Where

\[
\delta^2 U_i(t) = U_{i+1}(t) - 2U_i(t) + U_{i-1}(t),
\]

\[
\beta_i > 0, \varphi_i > 0
\]

\[
\delta^2 U_i(t) = \frac{2U_{i+1}(t) - 2U_i(t)}{h^2}, \quad \delta^2 U_i(t) = \frac{2U_{i+1}(t) - 2U_i(t)}{h^2}. \]

(10)

where \( \beta_i \) and \( \varphi_i \) are approximations of \( c(x_i) \) and \( u_0(x_i) \), respectively. There is another motivation which has cited our choice, one may remark our scheme, we have not chosen \( \beta_i = c(x_i) \) and \( \varphi_i = u_0(x_i) \). The motivation to give the initial data and the potential of this manner is two fold. Firstly, in a lot of situations, it is difficult to have either the exact value of the potential or that of the initial data. It is the case when one of there is, for instance, the solution of a complicated ordinary differential equation. Secondly we want to study the behavior of the quenching time when one perturbs slightly either the potential or the initial datum.

Here \((0, T_q^h)\) is the maximal time interval on which

\[
\| U_i(t) \|_{\infty} = \min_{0 \leq s \leq t} U_i(t).
\]

When the time \( T_q^h \) is finite, we say that the solution \( U_i(t) \) of (7)-(9) quenches in a finite time and the time \( T_q^h \) is called the quenching time of the solution \( U_i(t) \).

Definition 2.1

We say that the \( U_i(t) \) solution of (7)-(9) quenches in a finite time if there exist a finite time \( T_q^h \) such that \( U_{\text{min}}(t) > 0 \) for \( t \in (0, T_q^h) \) but \( \lim_{t \to T_q^h} U_{\text{min}}(t) = 0 \), where \( U_{\text{min}}(t) = \min_{0 \leq s \leq t} U_i(t) \).
The time $T^h_q$ is called the quenching time of the solution $U_h(t)$.

The following lemma is a semidiscrete form of the maximum principle.

**Lemma 2.1**

Let $\alpha_h(t) \in C^0([0,T], \mathbb{R})$ and let $V_h \in C^1([0,T], \mathbb{R})$ be such that

$$
\frac{dV(t)}{dt} - \alpha_k(t) V_k(t) \geq 0, \quad 0 \leq k \leq I, \quad t \in (0,T),
$$

(11)

Then $V_i(t) \geq 0, \quad 0 \leq i \leq I, \quad t \in (0,T)$. 

**Proof:**

Let $T_0$ be any quantity satisfying the inequality $T_0 < T$ and define the vector $Z_h(t) = e^{\lambda V_h(t)}$ where $\lambda$ is such that

$$\alpha_i(t) - \lambda > 0, \quad 0 \leq i \leq I, \quad t \in [0,T_0].$$

Set $m = \min_{0 < t \leq T_0} \| Z_h(t) \|_{\infty}$. Since $Z_h(t)$ is a continuous vector on the compact $[0,T_0]$ there exist $i_0 \in \{0,...,I\}$ and $t_0 \in [0,T_0]$ such that $m = Z_{i_0}(t_0)$. We observe that

$$
\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0, \quad (13)
$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \geq 0.
$$

(14)

Since $Z_{i_0}(t_0) \leq Z_{i_0}(t_0 - k), Z_{i_0+1}(t_0) \geq Z_{i_0}(t_0)$, and $Z_{i_0-1}(t_0) \geq Z_{i_0}(t_0)$. From (11), we obtain the following inequality

$$
\frac{dZ_{i_0}(t_0)}{dt} = \frac{dZ_{i_0}(t_0)}{dt} - \alpha_k(t_0) + (\alpha_k(t_0) - \lambda)Z_{i_0}(t_0) \geq 0.
$$

(15)

We deduce from (13)-(15) that $(\alpha_k(t_0) - \lambda)Z_{i_0}(t_0) \geq 0$, which implies that $Z_{i_0}(t_0) \geq 0$. Therefore, $V_h(t) \geq 0$ for $t \in [0,T_0]$ and the proof is complete.

Another form of the maximum principle for semidiscrete equations is the following comparison lemma.

**Lemma 2.2**

Let $f \in C^0([0,\mathbb{R}])$. If $V_h, W_h \in C^1([0,T], \mathbb{R})$ are such that

$$
\frac{dV(t)}{dt} - \delta^2 V(t) + f(V(t),t) < 0, \quad 0 \leq t \leq T,
$$

$$
\frac{dW(t)}{dt} - \delta^2 W(t) + f(W(t),t), \quad 0 \leq t \leq T,
$$

then $V(t) < W(t), \quad 0 \leq t \leq T$.

**Proof:**

Let $Z_h(t) = W_h(t) - V_h(t)$ and let $t_0$ be the first $t \in (0,T)$ such that $Z_h(t_0) > 0$ for $t \in (0,t_0)$ but $Z_h(t_0) = 0$ for a certain $i_0 \in \{0,...,I\}$. We see that

$$
\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0,
$$

$$\delta^2 Z_{i_0}(t_0) \geq 0.$$

Therefore, we have

$$
\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + f(W_{i_0}(t_0),t_0) - f(V_{i_0}(t_0),t_0) \leq 0,
$$

which contradicts the first strict inequality of the lemma and this ends the proof.

**Lemma 2.3**

Let $U_h$ be the solution of (7)-(9). Assume that the initial data at (9) satisfies $\varphi_i < 1, 1 \leq i \leq I$. Then, we have $U_i(t) < 1, \quad t \in (0,T^h_q)$. 

**Proof:**

Let $t_0 \in (0,T^h_q)$ be the first time $t \in (0,T^h_q)$ such that $U_{i_0}(t_0) < 1$, for $1 \leq i_0 \leq I$; $t \in (0,t_0)$, such that $U_{i_0}(t_0) < 1$, for $1 \leq i_0 \leq I$; $t \in (0,t_0)$, but $U_{i_0}(t_0) = 1$ for certain $j \in \{1,...,I-1\}$. We have

$$
\frac{dU_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{U_{i_0}(t_0) - U_{i_0}(t_0 - k)}{k} \geq 0
$$

and

$$
\delta^2 U_{i_0}(t_0) = \frac{U_{i_0+1}(t_0) - 2U_{i_0}(t_0) + U_{i_0-1}(t_0)}{h^2}.
$$

Which implies that

$$
\frac{dU_{i_0}(t_0)}{dt} - \delta^2 U_{i_0}(t_0) + \beta U_{i_0}(t_0) > 0.
$$

But, this contradicts (7) and the proof is complete.
3- Semi discrete quenching solutions

In this section, under some assumptions, we show that the solution $U_h$ of (7)-(9) quenches in a finite time and estimate its semi-discrete quenching time.

We need the following result about the operator $\delta^2$.

**Lemma 3.1**

Let $U_h \in \mathbb{R}^{|I|+1}$ be such that $U_h > 0$. Then, we have

$$\delta^2(f(U)) \geq f'(U_i)\delta^2 U_i, \quad 0 \leq i \leq I.$$  

**Proof:**

Applying Taylor's expansion, we find that

$$\delta^2(f(U))_i = f'(U_i)\delta^2 U_i + \frac{(U_{i+1} - U_i)^2}{h^2} f''(\theta_i)$$

where $\theta_i$ is an intermediate value between $U_i$ and $U_{i+1}$, $\eta_i$ the one between $U_{i-1}$ and $U_i$, $U_{i-1} = U_1, U_{i+1} = U_I - 1, \eta_0 = \theta_0$, $\eta_I = \theta_I$. Use the fact that $U_h > 0$ to complete the rest of the proof. 

The statement of the result about solutions which quench in a finite time is the following.

**Theorem 3.1**

Let $U_h$ be the solution of (7)-(9) and assume that there exists a positive constant $A \in (0,1]$ and the initial data at (9) satisfies

$$\delta^2 \varphi_i - \beta_i f(\varphi_i) \leq -Af(\varphi_i), 0 \leq i \leq I.$$  

(16)

Then, the solution $U_h$ quenches in a finite time $T_q^h$ and we have the following estimate

$$T_q^h \leq \frac{1}{A} \int_0^U \frac{d\sigma}{f(\sigma)}.$$  

**Proof:**

Since $(0,T_q^h)$ is the maximal time interval on which $\|U_h(t)\|_{\|\|} > 0$, our aim is to show that $T_q^h$ is finite and satisfies the above inequality.

Introduce the vector $J_h(t)$ defined as follows

$$J_h(t) = \frac{dU_h(t)}{dt} + Af(U_h(t)), \quad 0 \leq i \leq I.$$  

A straightforward calculation gives

$$\frac{dJ_h(t)}{dt} - \delta^2 J_h(t) \leq \frac{dU_h(t)}{dt} (\frac{dU_h(t)}{dt} - \delta^2 U_h(t)) + Af'(U_h(t)) \frac{dU_h(t)}{dt} - Af(U_h(t))$$

$$\leq \beta_i f'(U_i) J_i, 0 \leq i \leq I.$$  

Using (7), we arrive at

$$\frac{dJ_h(t)}{dt} - \delta^2 J_i \leq \beta_i f'(U_i) J_i,$$

$$0 \leq i \leq I, \quad t \in (0, T_q^h).$$

From (16), we observe that $J_h(0) \leq 0$. We deduce from Lemma 2.1 that $J_h(t) \leq 0$ for $t \in (0, T_q^h)$, which implies that

$$\frac{dU_h(t)}{dt} \leq -Af(U_h(t)), \quad 0 \leq i \leq I, \quad t \in (0, T_q^h).$$

These estimates may be rewritten in the following form

$$\frac{dU_h(t)}{dt} \leq -Adt, 0 \leq i \leq I.$$  

Integrating the above inequalities over the interval $(t,T_q^h)$, we get

$$T_q^h - t \leq \frac{1}{A} \int_t^{T_q^h} \frac{d\sigma}{f(\sigma)}, \quad 0 \leq i \leq I.$$  

Using the fact that $\|\varphi_h\|_{\|\|} = U_q_h(0)$ for a certain $i_q \in \{0,...,I\}$ and taking $t = 0$ in (18), we obtain the desired result. 

**Remark 3.1**

The inequalities (18) imply that

$$T_q^h - t_0 \leq \frac{1}{A} \int_{t_0}^{T_q^h} \frac{d\sigma}{f(\sigma)} \quad \text{for } t_0 \in (0, T_q^h),$$

and
\[ \| U_h(t) \|_{\inf} \geq H(A(T_q^h - t)) \text{ for } t \in (0, T_q^h), \]

where \( H(s) \) is the inverse of the function

\[ F(s) = \int_0^s \frac{d\sigma}{f(\sigma)}. \]

**Remark 3.2**

Let \( U_h \) be the solution of (7)-(9). Then, we have

\[ T_q^h \geq \frac{1}{\| \beta_h \|_{\infty}} \int_0^{\sqrt{1}} d\sigma, \]

and

\[ \| U_h(t) \|_{\inf} \leq \beta_h \| H(A(T_q^h - t)) \| \text{ for } t \in (0, T_q^h). \]

To prove these estimates, we proceed as follows. Introduce the function \( v(t) \) defined as follows

\[ v(t) = \| U_h(t) \|_{\inf} \text{ for } t \in [0, T_q^h). \]

Then, there exist \( t_1, t_2 \in [0, 1] \) such that \( v(t_1) = U(t_1) \) and \( v(t_2) = U(t_2) \). We observe that

\[ v(t_2) - v(t_1) \geq U(t_1) - U(t_2) = dU_{t_2} - dU_{t_1} = (t_2 - t_1) \frac{dU(t_2)}{dt} + o(t_2 - t_1), \]

\[ v(t_2) - v(t_1) \leq U(t_1) - U(t_2) = dU_{t_1} - dU_{t_2} = (t_2 - t_1) \frac{dU(t_1)}{dt} + o(t_2 - t_1), \]

which implies that \( v(t) \) is Lipschitz continuous. Further, if \( t_2 > t_1 \), then

\[ \frac{v(t_2) - v(t_1)}{t_2 - t_1} \geq \frac{dU_{t_2}}{dt} + o(1) = \delta^2 U_{t_2} - \beta_{t_2} f(U_{t_2}) + o(1). \]

Obviously,

\[ \delta^2 U_{t_2} \geq 0. \]

Letting \( t_1 \to t_2 \), and using the fact that \( \beta_{t_1} \leq \| B_h \|_{\infty} \), we obtain

\[ \frac{dv(t)}{dt} \geq - \| \beta_h \|_{\infty} f(v(t)) \text{ for } t \in (0, T_q^h) \]

Or equivalently

\[ \frac{dv}{f(v(t))} \geq - \| \beta_h \|_{\infty} dt \text{ for } t \in (0, T_q^h). \]

Integrate the above inequality over \((t, T_q^h)\) to obtain \( T_q^h \geq \frac{1}{\| \beta_h \|_{\infty}} \int_v(t) d\sigma \).

Since \( v(t) = \| U_h(t) \|_{\inf} \), we arrive at

\[ T_q^h \leq \frac{1}{\| \beta_h \|_{\infty}} \int_0^v(T_q^h) d\sigma \]

and the second estimate follows. To obtain the first one, it suffices to replace \( t \) by 0 in the above inequality and use the fact that \( \| \phi_h \|_{\inf} = \| U_h(0) \|_{\inf} \).

**Remark 3.3**

If \( \varphi = \alpha, 0 \leq i \leq I \), where \( \alpha \) is a positive constant, then one may take \( A = 1 \). It may imply that the potential equals to 1. In this case,

\[ T_q^h = \frac{\alpha p + 1}{p + 1} \text{ and } \| U_h(t) \|_{\inf} = (\alpha p + 1)(T_q^h - t)^{\frac{1}{p + 1}} \]

for \( t \in (0, T_q^h) \).

**4- Convergence of the semidiscrete quenching time**

In this section, under some assumptions, we show that the solution of the semidiscrete problem quenches in a finite time and its semidiscrete quenching time converges to the real one when the mesh size goes to zero. We denote

\[ c_h = (c(x_0), \ldots, c(x_I))^T, u_h(t) = (u(x_0, t), \ldots, u(x_I, t))^T \]

and \( \| U_h(t) \|_{\infty} = \max_{0 \leq i \leq I} |U_i(t)| \).

In order to obtain the convergence of the semidiscrete quenching time, we firstly prove the following theorem about the convergence of the semidiscrete scheme.

**Theorem 4.1**

Assume that the problem (1)-(3) has a solution \( u \in C^{1,1}([0,1] \times [0, T]) \) such that

\[ \min_{t \in [0, T]} u_{\min}(t) = \bar{u} > 0. \]

Suppose that the potential at (7) and the initial data at (9) satisfy

\[ \| \phi_h - u_h(0) \|_{\infty} = o(1) \quad \text{as} \quad h \to 0, \]

\[ \| \beta_h - c_h \|_{\infty} = o(1) \quad \text{as} \quad h \to 0. \]
Then, for \( h \) sufficiently small, the problem (7)-(9) has a unique solution \( U_h \in C^1([0,T],\mathbb{R}^{l+1}) \) such that the following relation holds

\[
\max_{\|u\|_x} \| \beta_h - c_h \|_x = 0(\| \phi_h - u_h(0) \|_x + h^2) \quad \text{as} \; h \to 0.
\]

**Proof:**

Let \( K > 0 \) and \( L > 0 \) be such that

\[
\frac{\|u_{\infty} \|_x}{2} \leq K, \quad f(P) \leq K \quad \text{and} \quad -\left( \| c_h \|_x + 1 \right) f'(P) \leq L.
\]

The problem (7)-(9) has for each \( h \), a unique solution \( U_h \in C^1([0,T^h],\mathbb{R}^{l+1}) \) . Let \( t(h) \leq \min\{T,T^h\} \) be the greatest value of \( t > 0 \) such that

\[
\| U_h(t) - u_h(t) \|_x < \frac{\hat{h}}{2} \quad \text{for} \; t \in (0,t(h)). \tag{22}
\]

The relation (19) implies that \( t(h) > 0 \) for \( h \) sufficiently small. By the triangle inequality, we obtain

\[
\| U_h(t) \|_x \geq \| u_h(t) \|_x - \| U_h(t) - u_h(t) \|_x \quad \text{for} \; t \in (0,t(h)), \quad \text{which implies that}
\]

\[
\| U_h(t) \|_x \geq \frac{\hat{h}}{2} - \frac{\hat{h}}{2} \quad \text{for} \; t \in (0,t(h)). \tag{23}
\]

Since \( u \in C^{1,1} \), taking the derivative in \( x \) on both sides of (1) and due to the fact that \( u_x, u_x \) vanish at \( x = 0 \) and \( x = 1 \), we observe that \( u_{\infty} \) also vanishes at \( x = 0 \) and

\[
x = 1 . \quad \text{Applying Taylor's expansion, we discover that}
\]

\[
u_x(x_1,t) = \delta^2 u(x_1,t) - \frac{h^2}{12} u_{\infty\infty}(x_1,t), \quad 0 \leq i \leq 1, \quad t \in (0,t(h)).
\]

To establish the above equalities for \( i = 0 \) and \( i = 1 \), we have used the fact that \( u_x \) and \( u_{\infty\infty} \) vanish at \( x = 0 \) and \( x = 1 \). A direct calculation yields

\[
u_{xx}(x_1,t) - \delta^2 u_{xx}(x_1,t) = -\beta_i f(u(x_1,t)) - \frac{h^2}{12} u_{\infty\infty\infty}(x_1,t) + (\beta_i - c(x_1)f(u(x_1,t)), \quad 1 \leq i \leq l - 1.
\]

Let \( e_h(t) = U_h(t) - u_h(t) \) be the error of discretization. From the mean value theorem, we have

\[
\frac{de_i(t)}{dt} - \delta^2 e_i(t) = -\beta_i f(\theta_i) e_i + \frac{h^2}{12} u_{\infty\infty\infty}(x_i,t) + (\beta_i - c(x_1)f(u(x_1,t)), \quad 0 \leq i \leq l, t \in (0,t(h)),
\]

where \( \theta_i \) is an intermediate value between

\[
U_i(t) \quad \text{and} \quad u(x_i,t) . \quad \text{Using (21), (22), we arrive at}
\]

\[
\frac{de_i(t)}{dt} - \delta^2 e_i(t) \leq L | e_i(t) | + Kh^2 + K \| \beta_h - c_h \|_x, \quad 0 \leq i \leq l, t \in (0,t(h)).
\]

Introduce the vector \( z_h(t) \) defined as follows

\[
z_h(t) = e^{(L+1)t}(\| \phi_h - u_h(0) \|_x + Kh^2 + K \| \beta_h - c_h \|_x), \quad 0 \leq i \leq l, t \in (0,t(h)).
\]

A straightforward computation reveals that

\[
\frac{dz_i}{dt} - \delta^2 z_i > L | z_i | + Kh^2 + K \| \beta_h - c_h \|_x, \quad 0 \leq i \leq l, t \in (0,t(h)),
\]

\[
z_h(0) > e_i(0), 0 \leq i \leq l.
\]

It follows from Comparison Lemma 2.2 that

\[
z_h(0) > e_i(0) \quad \text{for} \; t \in (0,t(h)), \quad 0 \leq i \leq l.
\]

In the same way, we also prove that

\[
z_h(0) > -e_i(t) \quad \text{for} \; t \in (0,t(h)), \quad 0 \leq i \leq l,
\]

which implies that

\[
\| U_h(t) - u_h(t) \|_x \leq e^{(L+1)t}(\| \phi_h - u_h(0) \|_x + Kh^2 + K \| \beta_h - c_h \|_x), \quad t \in (0,t(h)).
\]

Let us show that \( t(h) = \min\{T,T^h\} \) . Suppose that

\[
t(h) < \min\{T,T^h\}. \quad \text{From (22), we obtain}
\]

\[
\frac{\hat{h}}{2} \leq \| U_h(t(h)) - u_h(t(h)) \|_x \leq e^{(L+1)t}(\| \phi_h - u_h(0) \|_x + Kh^2 + K \| \beta_h - c_h \|_x).
\]

Let us notice that both last formulas for \( t(h) \) are valid for sufficiently small \( h \) . Since the term on the right hand side of the above inequality goes to zero as \( h \) goes to zero, we deduce that \( \frac{\hat{h}}{2} \leq 0 \), which is
impossible. Consequently \( t(h) = \min\{T, T^h\} \). Now, let us show that \( t(h) = T \).

Suppose that \( t(h) = T^h < T \). Reasoning as above, we prove that

we have a contradiction and the proof is complete.

Now, we are in a position to prove the main theorem of this section.

**Theorem 4.2**

Suppose that the problem (1)-(3) has a solution \( u \) which quenches in a finite time \( T_q \) such that \( u \in C^4(I) \). Assume that the potential at (7) and the initial data at (9) satisfy the conditions (19) and (20), respectively. Under the hypothesis of Theorem 3.1, the problem (7)-(9) has a solution \( U^h \) which quenches in a finite time \( T^h \) and we have \( \lim_{h \to 0} T^h = T_q \).

**Proof:**

Let \( 0 < \varepsilon < T_q / 2 \). There exists \( \bar{n} \in (0,1) \) such that

\[
\frac{1}{\Lambda} \int_0^{\bar{n}} \frac{d\sigma}{f(\sigma)} \leq \frac{\varepsilon}{2}.
\]

Since \( u \) quenches in a finite time \( T_q \), there exist \( h_0(\varepsilon) > 0 \) and a time \( T_0 \in (T_q - \frac{\varepsilon}{2}, T_q) \) such that

\[
0 < u_{\min}(t) < \frac{n}{2} \quad \text{for} \ t \in [T_0, T_q], \quad h \leq h_0(\varepsilon).
\]

It is not hard to see that

\[
\min_{t \in [T_0, T_q]} u(t) > 0 \quad \text{for} \ t \in [0, T_0], \quad h \leq h_0(\varepsilon).
\]

From Theorem 4.1, the problem (7)-(9) has a solution \( U^h(t) \) and we get \( \| U^h(t) - u_h(t) \|_\infty \leq \bar{n} \)

for \( t \in [0, T_0] \), \( h \leq h_0(\varepsilon) \), which implies that

\[
\| U^h(T_0) - u_h(T_0) \|_\infty \leq \bar{n} \quad \text{for} \ h \leq h_0(\varepsilon).
\]

Applying the triangle inequality, we find that

\[
\| U^h(T_0) \|_\infty \leq \| U^h(T_0) - u_h(T_0) \|_\infty + \| u_h(T_0) \|_\infty \leq \frac{\bar{n} + \bar{n}}{2} = \bar{n}
\]

for \( h \leq h_0(\varepsilon) \).

From Theorem 3.1, \( U^h(t) \) quenches at the time \( T^h_q \). We deduce from Remark 3.1 and (22) that for \( h \leq h_0(\varepsilon) \)

\[
|T^h_q - T_q| \leq |T^h_q - T_0| + |T_0 - T_q| \leq \frac{1}{\Lambda} \int_{T_0}^{T_h} \frac{d\sigma}{f(\sigma)} + \frac{\varepsilon}{2} \leq \varepsilon
\]

which leads us to the desired result.

5- Full discretizations

In this section, we study the phenomenon of quenching using a full discrete explicit scheme of (1)-(3). Approximate the solution \( u(x,t) \) of the problem (1)-(3) by the solution \( U_t^{(n)} = (U_t^{(n)}, U_t^{(n)}, \ldots, U_t^{(n)})^T \) of the following explicit scheme

\[
\frac{U_t^{(n+1)} - U_t^{(n)}}{\Delta t} = \frac{f(U_t^{(n)})}{\| U_t^{(n)} \|_\infty}.
\]

where \( n \geq 0 \),

\[
\frac{f(s)}{s} - f'(s) \leq 0, \quad \text{for} \ s > 0.
\]

**Theorem 4.5**

\[U_t^{(n+1)} \geq U_t^{(n)} + (1 - \frac{2}{\Delta t} \frac{\beta_n}{\Delta_t} \| U_t^{(n)} \|_\infty) U_t^{(n)},\]

\[U_t^{(i)} \geq U_t^{(i-1)} + (1 - \frac{2}{\Delta t} \frac{\beta_n}{\Delta_t} \| U_t^{(i)} \|_\infty) U_t^{(i-1)}, \quad 1 \leq i \leq I-1.
\]

In order to permit the discrete solution to reproduce the properties of the continuous one when the time \( t \) approaches the quenching time \( T_q \), we need to adapt the size of the time step so that we choose
\[ \Delta t_n = \min\left\{ \frac{(1-\tau)h^2}{2}, \tau \frac{f(\|U_h^{(n)}\|_{\infty})}{\|U_h^{(n)}\|_{\text{inf}}} \right\} \]

With \( 0 < \tau < 1 \). We observe that
\[ 1 - 2 \frac{\Delta t_n}{h^2} - \beta_n \|a_h^{(n)}\|_{\infty} \Delta t_n \frac{f(\|U_h^{(n)}\|_{\infty})}{\|U_h^{(n)}\|_{\text{inf}}} \geq 0, \]

which implies that \( U_h^{(n+1)} > 0 \). Thus, since by hypothesis \( U_h^{(0)} = \varphi_h > 0 \), if we take \( \Delta t_n \) as defined above, then using a recursion argument, we see that the positivity of the discrete solution is guaranteed. Here, \( \tau \) is a parameter which will be chosen later to allow the discrete solution \( U_h^{(n)} \) to satisfy certain properties useful to get the convergence of the numerical quenching time defined below. If necessary, we may take
\[ \Delta t_n = \min\left\{ \frac{(1-\tau)h^2}{K}, \tau \frac{f(\|U_h^{(n)}\|_{\infty})}{\|U_h^{(n)}\|_{\text{inf}}} \right\} \]

with \( K > 2 \) because in this case, the positivity of the discrete solution is also guaranteed. The following lemma is a discrete form of the maximum principle.

**Lemma 5.1**

Let \( a_h^{(n)} \) and \( V_h^{(n)} \) be two sequences such that \( a_h^{(n)} \) is bounded and
\[ \delta_i V_i^{(n)} - \delta^2 V_i^{(n)} + a_i^{(n)} V_i^{(n)} \geq 0, \quad 0 \leq i \leq I, \quad n \geq 0, \] (27)
\[ V_i^{(0)} \geq 0, \quad 0 \leq i \leq I. \] (28)

Then \( V_i^{(n)} \geq 0 \) for \( n \geq 0 \), \( \forall \); \( 0 \leq i \leq I \)

if \( \Delta t_n \leq \frac{h^2}{2 + \|a_h^{(n)}\|_{\infty} h^2} \).

**Proof:**

If \( V_h^{(n)} \geq 0 \), then a routine computation yields
\[ V_i^{(n+1)} \geq \frac{2\Delta t_n}{h^2} V_i^{(n)} + (1 - 2 \frac{\Delta t_n}{h^2} - \Delta t_n \|a_h^{(n)}\|_{\infty}) V_i^{(n)}, \]
\[ V_i^{(n+1)} \geq \frac{\Delta t_n}{h^2} V_i^{(n)} + (1 - 2 \frac{\Delta t_n}{h^2} - \Delta t_n \|a_h^{(n)}\|_{\infty}) V_i^{(n)} + \frac{\Delta t_n}{h^2} V_i^{(n)}, \]
\[ 1 \leq i \leq I - 1, \]
\[ V_i^{(n+1)} \geq \frac{2\Delta t_n}{h^2} V_i^{(n)} + (1 - 2 \frac{\Delta t_n}{h^2} - \Delta t_n \|a_h^{(n)}\|_{\infty}) V_i^{(n)}. \]

Since \( \Delta t_n \leq \frac{h^2}{2 + \|a_h^{(n)}\|_{\infty} h^2} \),

we see that \( t_n \|a_h^{(n)}\|_{\infty} \)

is nonnegative. From (27), we deduce by induction that \( V_h^{(n)} \geq 0 \) which ends the proof. \( \square \)

A direct consequence of the above result is the following comparison lemma. Its proof is straightforward.

**Lemma 5.2**

Let \( V_h^{(n)}, W_h^{(n)} \) and \( a_h^{(n)} \) be three sequences such that \( a_h^{(n)} \) is bounded and
\[ \delta_i V_i^{(n)} - \delta^2 V_i^{(n)} + a_i^{(n)} V_i^{(n)} \leq \delta_i W_i^{(n)} - \delta^2 W_i^{(n)} + a_i^{(n)} W_i^{(n)}, \quad 0 \leq i \leq I, \quad n \geq 0, \]
\[ V_i^{(0)} \leq W_i^{(0)}, \quad 0 \leq i \leq I. \]

Then \( V_i^{(n)} \leq W_i^{(n)} \) for \( n \geq 0 \), \( \forall \); \( 0 \leq i \leq I \)

\[ \Delta t_n \leq \frac{h^2}{2 + \|a_h^{(n)}\|_{\infty} h^2}. \]

Now, let us give a property of the operator \( \delta \) stated in the following lemma. Its proof is quite similar to that of Lemma 3.1, so we omit it here.

**Lemma 5.3**

Let \( U^{(n)} \) be such that \( U^{(n)} > 0 \) for \( n \geq 0 \). Then, we have
\[ \delta_i f(U^{(n)}) \geq f'(U^{(n)}) \delta_i U^{(n)}, \quad n \geq 0. \]

**Lemma 5.4**

Let \( a, b \) be tow positive numbers such that \( b < 1 \)

then following estimate holds
\[ \sum_{n=0}^{\infty} \frac{ab^n}{f(ab^n)} \leq \frac{a}{f(a)} - \frac{1}{\ln(b)} \int_a^a \frac{d\sigma}{f(\sigma)}. \]

**Proof:** We have
\[ \int_0^\infty \frac{ab^x}{f(ab^x)} = \sum_{n=0}^{\infty} \int_n^{n+1} \frac{ab^x dx}{f(ab^x)}. \]

We observe that \( ab^x \geq ab^{x+1} \) for \( n \leq x \leq n+1 \),

which that \( \int_n^{n+1} \frac{ab^x dx}{f(ab^x)} \geq \frac{ab^{n+1}}{f(ab^{n+1})}. \)

Consequently, we get
\[
\int_a^b \frac{ab^x}{f(ab^x)} dx \geq \sum_{n=0}^{m} \int_a^b \frac{ab^x}{f(ab^x)} = -a \frac{f(a)}{f(x)} + \sum_{n=0}^{m} \frac{ab^x}{f(ab^x)}.
\]

Use the fact that
\[
\int_0^\infty \frac{ab^x}{f(ab^x)} dx = -\frac{1}{\ln(b)} \int_0^\infty f(\sigma) d\sigma
\]

to complete the rest of the proof. \|

The theorem below is the discrete version of Theorem 4.1.

**Theorem 5.1**

Suppose that the problem \((1) - (3)\) has a solution \(u \in C^4([0,1])\) such that
\[
\min_{t \in (0,T)} u(x,t) = \rho > 0.
\]
Assume that the initial data at \((26)\) satisfies the condition \((16)\). Then, the problem \((25)-(26)\) has a solution \(U_h^{(n)}\) for \(h\) sufficiently small, \(0 \leq n \leq J\) and the following relation holds
\[
\max_{0 \leq n \leq J} ||U_h^{(n)} - u_h^n|| = O(\|\varphi_h - u_h^0\|_\infty + \|c_h - \beta_h\|_\infty + h^2) \quad \text{as} \quad h \to 0.
\]

where \(J\) is any quantity satisfying the inequality
\[
\sum_{n=0}^{J} \Delta t_n \leq T \quad \text{and} \quad t_n = \sum_{j=0}^{n} \Delta t_j.
\]

**Proof:**

For each \(h\), the problem \((25)-(26)\) has a solution \(U_h^{(n)}\). Let \(N \leq J\) be the greatest value of \(n\) such that
\[
||U_h^{(n)} - u_h^n||_\infty < \frac{\rho}{2} \quad \text{for} \quad n < N. \quad (29)
\]

We know that \(N \geq 1\) because of \((16)\). Applying the triangle inequality, we have
\[
||U_h^{(n)}||_\infty \geq ||u_h^n||_\infty - ||U_h^{(n)} - u_h^n||_\infty \geq \frac{\rho}{2} \quad \text{for} \quad n < N. \quad (30)
\]

As in the proof of Theorem 4.1, using Taylor's expansion, we find

\[
\delta \varphi_h - \varphi_{x} (x,t_n) = -\delta \varphi_h - \varphi_{x} (x,t_n) + \beta f(u(x,t_n)) + (c(x) - \beta) f(u(x,t_n)) = \frac{h^2}{12} u_{xxxx}(x,t_n) + \frac{\Delta t_n}{2} u_x(x,t_n).
\]

Let \(e_h^{(n)} = U_h^{(n)} - u_h^n(t_n)\) be the error of discretization. From the mean value theorem, we get for \(n < N, \quad 0 \leq i \leq I\),
\[
\delta \varphi_h - \varphi_{x} (x,t_n) = -\delta \varphi_h - \varphi_{x} (x,t_n) + \beta f(u(x,t_n)) + (c(x) - \beta) f(u(x,t_n)) = \frac{h^2}{12} u_{xxxx}(x,t_n) + \frac{\Delta t_n}{2} u_x(x,t_n).
\]

A straightforward computation gives
\[
\delta \varphi_h - \varphi_{x} (x,t_n) > \beta f'(\xi_h^{(n)}) e_h^{(n)} + M \|c_h - \beta_h\|_\infty + \Delta h^2, \quad \text{for} \quad 0 \leq i \leq I, n < N.
\]

As in the proof of Theorem 4.1, using Taylor's expansion, we find

\[
\delta \varphi_h - \varphi_{x} (x,t_n) = -\delta \varphi_h - \varphi_{x} (x,t_n) + \beta f(u(x,t_n)) + (c(x) - \beta) f(u(x,t_n)) = \frac{h^2}{12} u_{xxxx}(x,t_n) + \frac{\Delta t_n}{2} u_x(x,t_n).
\]

Let \(e_h^{(n)} = U_h^{(n)} - u_h^n(t_n)\) be the error of discretization. From the mean value theorem, we get for \(n < N, \quad 0 \leq i \leq I\),
\[
\delta \varphi_h - \varphi_{x} (x,t_n) = -\delta \varphi_h - \varphi_{x} (x,t_n) + \beta f(u(x,t_n)) + (c(x) - \beta) f(u(x,t_n)) = \frac{h^2}{12} u_{xxxx}(x,t_n) + \frac{\Delta t_n}{2} u_x(x,t_n).
\]

A straightforward computation gives
\[
\delta \varphi_h - \varphi_{x} (x,t_n) > \beta f'(\xi_h^{(n)}) e_h^{(n)} + M \|c_h - \beta_h\|_\infty + \Delta h^2, \quad \text{for} \quad 0 \leq i \leq I, n < N.
\]

As in the proof of Theorem 4.1, using Taylor's expansion, we find

\[
\delta \varphi_h - \varphi_{x} (x,t_n) = -\delta \varphi_h - \varphi_{x} (x,t_n) + \beta f(u(x,t_n)) + (c(x) - \beta) f(u(x,t_n)) = \frac{h^2}{12} u_{xxxx}(x,t_n) + \frac{\Delta t_n}{2} u_x(x,t_n).
\]
Let us show that \( N = J \). Suppose that \( N < J \). If we replace \( n \) by \( N \) in (29) and use (30), we find that

\[
\frac{\rho}{2} \leq \| U_h^{(N)}(t) - u_h(t) \|_{\infty} \leq e^{(1+\tau)T} (\| \phi_h - u_h(0) \|_{\infty} + Mh^2 + M \| c_h - \beta_h \|_{\infty}).
\]

Since the term on the right hand side of the second inequality goes to zero as \( h \) goes to zero, we deduce that \( \frac{\rho}{2} \leq 0 \), which is a contradiction and the proof is complete. \( \square \) To handle the phenomenon of quenching for discrete equations, we need the following definition.

**Definition 5.1**

We say that the solution \( U_h^{(n)} \) of (25)-(26) quenches in a finite time if \( \| U_h^{(n)} \|_{\text{inf}} > 0 \) for \( n \geq 0 \), but

\[
\lim_{n \to +\infty} \| U_h^{(n)} \|_{\text{inf}} = 0 \quad \text{and} \quad T_h^{\text{N}} = \lim_{n \to +\infty} \sum_{i=0}^{n-1} \Delta t_i < \infty.
\]

The number \( T_h^{\text{N}} \) is called the numerical quenching time of \( U_h^{(n)} \). The following theorem reveals that the discrete solution \( U_h^{(n)} \) of (25)-(26) quenches in a finite time under some hypotheses.

**Theorem 5.2**

Let \( U_h^{(n)} \) be the solution of (25)-(26). Suppose that there exists a constant \( A \in (0,1] \) such that the initial data at (26) satisfies

\[
\delta^2 \phi_i - \beta \phi_i f(\phi_i) \leq -Af(\phi_i), \quad 0 \leq i \leq I. \quad (31)
\]

Then \( U_h^{(n)} \) is nonincreasing and quenches in a finite time \( T_h^{\text{N}} \) which satisfies the following estimate

\[
T_h^{\text{N}} \leq \frac{\| \phi_h \|_{\text{inf}}}{f(\| \phi_h \|_{\text{inf}})} - \frac{\tau}{\ln(1 - \tau)} \int_0^{\| \phi_h \|_{\text{inf}}} \frac{d\sigma}{f(\sigma)},
\]

where

\[
\tau' = A \min\left\{ \frac{(1-\tau)h^2 f(\| \phi_h \|_{\text{inf}})}{2 \| \phi_h \|_{\text{inf}}}, \tau \right\}.
\]

**Proof:**

Introduce the vector \( J_h^{(n)} \) defined as follows

\[
J_h^{(n)} = \delta U_h^{(n)} + Af(U_h^{(n)}), \quad 0 \leq i \leq I, \quad n \geq 0.
\]

A straightforward computation yields for \( 0 \leq i \leq I, n \geq 0 \),

\[
\partial_t J_h^{(n)} - \delta^2 J_h^{(n)} = \delta_t \left( \delta^2 U_h^{(n)} - \delta^2 U_h^{(n)} \right) + A\delta_t f(U_h^{(n)}) - A\delta^2 f(U_h^{(n)}).
\]

Using (25), we arrive at

\[
\partial_t J_h^{(n)} - \delta^2 J_h^{(n)} = -(\beta_i - A)\delta_t f(U_h^{(n)}) - A\delta^2 f(U_h^{(n)}),
\]

\( 0 \leq i \leq I, \quad n \geq 0 \).

It follows from Lemmas 5.3 and 3.1 that for \( 0 \leq i \leq I, n \geq 0 \),

\[
\partial_t J_h^{(n)} - \delta^2 J_h^{(n)} \leq -(\beta_i - A)\delta_t f(U_h^{(n)}) J_h^{(n)}, \quad 0 \leq i \leq I, n \geq 0.
\]

Obviously, the inequalities (31) ensure that \( J_h^{(n)} \leq 0 \). Applying Lemma 5.1, we get \( J_h^{(n)} \leq 0 \), for \( n \geq 0 \), which implies that

\[
U_h^{(n+1)} \leq U_h^{(n)} (1 - A\Delta t_n \frac{f(U_h^{(n)})}{U_h^{(n)}}), \quad (32)
\]

\( 0 \leq i \leq I, \quad n \geq 0 \).

These estimates reveal that the sequence \( U_h^{(n)} \) is nonincreasing. By induction, we obtain \( U_h^{(n)} \leq U_h^{(0)} = \phi_h \). Thus, the following holds

\[
A\Delta t_n \frac{f(\| U_h^{(n)} \|_{\text{inf}})}{\| U_h^{(n)} \|_{\text{inf}}} \geq A \min\left\{ \frac{(1-\tau)h^2 f(\| \phi_h \|_{\text{inf}})}{2 \| \phi_h \|_{\text{inf}}}, \tau \right\} = \tau'. \quad (33)
\]

Let \( i_0 \) be such that \( \| U_h^{(n)} \|_{\text{inf}} = U_h^{(n)} \). Replacing \( i \) by \( i_0 \), we obtain

\[
\| U_h^{(n+1)} \|_{\text{inf}} \leq A \| U_h^{(n)} \|_{\text{inf}} (1 - \tau'), \quad n \geq 0, \quad (34)
\]

and by iteration, we arrive at
\[ \| U_{h}^{(n)} \|_{\inf} \leq \| U_{h}^{(0)} \|_{\inf} (1 - \tau')^{n} = \| \varphi_{h} \|_{\inf} (1 - \tau')^{n}, \quad n \geq 0. \quad (35) \]

Since the term on the right hand side of the above equality goes to zero as \( n \) approaches infinity, we conclude that \( \| U_{h}^{(n)} \|_{\inf} \) tends to zero as \( n \) approaches infinity. Now, let us estimate the numerical quenching time. Due to (33) and the restriction \( \Delta t_{n} \leq \frac{\tau \| \varphi_{h} \|_{\inf}}{f(\| \varphi_{h} \|_{\inf})}, \) it is not hard to see that

\[ \Sigma_{n=0}^{\infty} \Delta t_{n} \leq \sum_{n=0}^{\infty} \frac{\| \varphi_{h} \|_{\inf} (1 - \tau')^{n}}{f(\| \varphi_{h} \|_{\inf})}. \]

because \( s = \frac{s}{f(s)} \) is nondecreasing for \( s > 0 \). It follows from Lemma 5.4 that

\[ \Sigma_{n=0}^{\infty} \Delta n_{t} \leq \frac{\tau \| \varphi_{h} \|_{\inf}}{f(\| \varphi_{h} \|_{\inf})} \int_{0}^{\infty} \frac{\sigma^{\| \varphi_{h} \|_{\inf}}}{f(\sigma)} d\sigma. \]

Use the fact that the quantity on the right hand side of the above inequality converges towards is finite to complete the rest of the proof. \[ \square \]

Remark 5.1 From (35), we deduce by induction that

\[ \| U_{h}^{(n)} \|_{\inf} \leq \| U_{h}^{(0)} \|_{\inf} (1 - \tau')^{n-1} \quad \text{for} \quad n \geq q, \]

and we see that

\[ T_{h}^{n} - t_{q} = \sum_{n=0}^{\infty} \Delta n_{t} \leq \sum_{n=0}^{\infty} \frac{\| U_{h}^{(q)} \|_{\inf} (1 - \tau')^{n-1}}{f(\| \varphi_{h} \|_{\inf})}, \]

because \( s = \frac{s}{f(s)} \) is nondecreasing for \( s > 0 \). It follows from Lemma 5.4 that

\[ T_{h}^{n} - t_{q} \leq \frac{\tau \| \varphi_{h} \|_{\inf}}{f(\| \varphi_{h} \|_{\inf})} \int_{0}^{\infty} \frac{\sigma^{\| \varphi_{h} \|_{\inf}}}{f(\sigma)} d\sigma. \]

Since

\[ \tau' = A \min\{ \frac{(1-\tau) h^{2} f(\| \varphi_{h} \|_{\inf})}{2 \| \varphi_{h} \|_{\inf}}, \tau \}, \]

if we take \( \tau = h^{2} \), we get

\[ \tau' = A \min\{ \frac{(1-\tau) h^{2} f(\| \varphi_{h} \|_{\inf})}{2 \| \varphi_{h} \|_{\inf}}, 1 \} \geq A \min\{ \frac{f(\| \varphi_{h} \|_{\inf})}{4 \| \varphi_{h} \|_{\inf}}, 1 \}. \]

Therefore, there exist constants \( c_{0}, c_{1} \) such that

\[ 0 \leq c_{0} \leq \tau / \tau' \leq c_{1} \] and

\[ \frac{-\tau}{\ln(1-\tau')} = O(1), \]

for the choice \( \tau = h^{2} \).

In the sequel, we take \( \tau = h^{2} \). Now, we are in a position to state the main theorem of this section.

**Theorem 5.3**

Suppose that the problem (1)-(3) has a solution \( u \) which quenches in a finite time \( T_{u} \) and \( u \in C^{1,2}([0,1] \times [0, T_{u}]). \) Assume that the initial data at \( T_{u} \) satisfies the condition (16). Under the assumption of Theorem 5.2, the problem (25)-(26) has a solution \( U_{h}^{(n)} \) which quenches in a finite time \( T_{h}^{n} \) and the following relation holds

\[ \lim_{h \to 0} T_{h}^{n} = T_{u}. \]

**Proof.**

We know from Remark 5.1 that \( \frac{\tau}{\ln(1-\tau')} \) is bounded. Letting \( 0 < \varepsilon < T_{u} / 2 \), there exists a constant \( R \in (0,1) \) such that

\[ \frac{\tau}{f(\sigma)} \int_{0}^{\infty} \frac{\sigma^{\| \varphi_{h} \|_{\inf}}}{f(\sigma)} d\sigma < \frac{\varepsilon}{2}. \]

Since \( u \) quenches at the time \( T_{u} \), there exist

\[ T_{1} \in (T_{u} - \frac{\varepsilon}{2}, T_{u}) \quad \text{and} \quad h_{0}(\varepsilon) > 0 \]

such that

\[ 0 < u_{\min}(t) < \frac{R}{2} \quad \text{for} \quad t \in [T_{1}, T_{u}], h \leq h_{0}(\varepsilon). \]

Let \( q \) be a positive integer such that \( t_{q} = \sum_{n=0}^{q-1} \Delta n_{t} \in [T_{1}, T_{u}) \)

for \( h \leq h_{0}(\varepsilon) \). It follows from Theorem 5.1 that

the problem (25) --(26) has a solution \( U_{h}^{(n)} \) which obeys

\[ \| U_{h}^{(n)} - u_{\tau}(t_{n}) \|_{\infty} < \frac{R}{2} \quad \text{for} \quad n \leq q, h \leq h_{0}(\varepsilon) \]

which implies that
From Theorem 5.2, \( U_h^{(n)} \) quenches at the time \( T_h^{\infty} \). It follows from Remark 5.1 and (36) that
\[
|T_h^{\infty} - t_q| \leq \tau \| U_h^{(n)} \|_{\infty} - \tau \int_{0}^{\ln(1 - \tau')} \frac{d\sigma}{f(\sigma)} < \frac{\varepsilon}{2}
\]
because \( \| U_h^{(n)} \|_{\infty} < R \) for \( h \leq h_0(\varepsilon) \). We deduce that
\[
h \leq h_0(\varepsilon), |T_q - T_h^{\infty}| \leq |T_q - t_q| + |t_q - T_h^{\infty}| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon,
\]
which leads us to the result. □

6- Numerical result

In this section, we give some computational experiments to the quenching time for the solution of the problem (1)-(3) to confirm the theory developed in the previous section. Firstly, we take the explicit scheme in (37)-(38).

\[
\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \delta^2 U_i^{(n)} - \beta_i(U_i^{(n)})^{-p}, 0 \leq i \leq I,
\]

(37)

\[
U_i^{(0)} = \varphi_i > 0, 0 \leq i \leq I,
\]

(38)

where \( n \geq 0, \Delta t_n = K \| U_h^{(n)} \|_{\infty}^{p+1} \) with \( K = h^2 \) and \( p = 1 \). In the case where,
\[
u_0(x) = \frac{0.99 + \varepsilon \sin(\pi x)}{2} \quad \text{with} \quad 0 < \varepsilon \leq 1 \quad \text{and} \quad \beta_i = 1 - 0.1 \varepsilon \sin(i\pi h).
\]

Secondly, we use the following implicit scheme in (39)-(40).

\[
\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \delta^2 U_i^{(n+1)} - \beta_i(U_i^{(n)})^{-p-1} U_i^{(n+1)},
\]

(39)

\[
0 \leq i \leq I,
\]

\[
U_i^{(0)} = \varphi_i > 0, 0 \leq i \leq I,
\]

(40)

In the case where, \( n \geq 0, \Delta t_n = K \| U_h^{(n)} \|_{\infty}^{p+1} \) with \( K = h^2, \varphi_i = \frac{2 + \varepsilon \sin(i\pi h)}{4}, 0 \leq i \leq I \)

and \( \beta_i = 1 - \varepsilon \sin(i\pi h) \). For the above implicit scheme, the existence and positivity of the discrete solution \( U_h^{(n)} \) is guaranteed using standard methods (see [6]). In the tables 1-10, in rows, we present the numerical quenching times, the numbers of iterations and the CPU times corresponding to meshes of 16, 32, 64, 128. We take for the numerical quenching time
\[
t_n = \sum_{j=0}^{n-1} \Delta t_j
\]
which is computed at the first time when
\[
\Delta t_n = \Delta t_{n+1} - t_n \leq 10^{-16}.
\]

Table 1:

<table>
<thead>
<tr>
<th>I</th>
<th>t_n</th>
<th>n</th>
<th>CPU time</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.312643</td>
<td>4161</td>
<td>1</td>
</tr>
<tr>
<td>32</td>
<td>0.312724</td>
<td>16023</td>
<td>3</td>
</tr>
<tr>
<td>64</td>
<td>0.312852</td>
<td>61257</td>
<td>60</td>
</tr>
<tr>
<td>128</td>
<td>0.312795</td>
<td>235525</td>
<td>1245</td>
</tr>
</tbody>
</table>

Table2: Numerical quenching times, numbers of iterations and CPU times (seconds) obtained with the explicit Euler method for \( \varepsilon = 1 \)

<table>
<thead>
<tr>
<th>I</th>
<th>t_n</th>
<th>n</th>
<th>CPU time</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.12683</td>
<td>3891</td>
<td>2</td>
</tr>
<tr>
<td>32</td>
<td>0.126653</td>
<td>14913</td>
<td>17</td>
</tr>
<tr>
<td>64</td>
<td>0.126611</td>
<td>56979</td>
<td>23</td>
</tr>
<tr>
<td>128</td>
<td>0.126600</td>
<td>217121</td>
<td>98</td>
</tr>
</tbody>
</table>

Table3: Numerical quenching times, numbers of iterations and CPU times (seconds) obtained with the implicit Euler method for \( \varepsilon = 100 \).
Table 4: Numerical quenching times, numbers of iterations and CPU times (seconds) obtained with the implicit Euler method for $\varepsilon = \frac{1}{100}$

<table>
<thead>
<tr>
<th>$I$</th>
<th>$t_n$</th>
<th>$n$</th>
<th>CPU time</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.127422</td>
<td>3901</td>
<td>2</td>
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<tr>
<td>32</td>
<td>0.126803</td>
<td>14928</td>
<td>11</td>
</tr>
<tr>
<td>64</td>
<td>0.126309</td>
<td>56979</td>
<td>33</td>
</tr>
<tr>
<td>128</td>
<td>0.126609</td>
<td>217134</td>
<td>109</td>
</tr>
</tbody>
</table>

Table 5: Numerical quenching times, numbers of iterations and CPU times (seconds) obtained with the implicit Euler method for $\varepsilon = \frac{1}{1000}$

<table>
<thead>
<tr>
<th>$I$</th>
<th>$t_n$</th>
<th>$n$</th>
<th>CPU time</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.125158</td>
<td>3887</td>
<td>2</td>
</tr>
<tr>
<td>32</td>
<td>0.125098</td>
<td>14864</td>
<td>16</td>
</tr>
<tr>
<td>64</td>
<td>0.124943</td>
<td>56779</td>
<td>222</td>
</tr>
<tr>
<td>128</td>
<td>0.124932</td>
<td>216314</td>
<td>3887</td>
</tr>
</tbody>
</table>

Table 6: Numerical quenching times, numbers of iterations and CPU times (seconds) obtained with the implicit Euler method for $\varepsilon = \frac{1}{10000}$

<table>
<thead>
<tr>
<th>$I$</th>
<th>$t_n$</th>
<th>$n$</th>
<th>CPU time</th>
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<td>16</td>
<td>0.125142</td>
<td>3887</td>
<td>2</td>
</tr>
<tr>
<td>32</td>
<td>0.125092</td>
<td>14864</td>
<td>11</td>
</tr>
<tr>
<td>64</td>
<td>0.124944</td>
<td>56778</td>
<td>133</td>
</tr>
<tr>
<td>128</td>
<td>0.125012</td>
<td>216314</td>
<td>98</td>
</tr>
</tbody>
</table>

Table 8: Numerical quenching times, numbers of iterations and CPU times (seconds) obtained with the implicit Euler method for $\varepsilon = \frac{1}{100000}$

<table>
<thead>
<tr>
<th>$I$</th>
<th>$t_n$</th>
<th>$n$</th>
<th>CPU time</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.125715</td>
<td>3903</td>
<td>21</td>
</tr>
<tr>
<td>32</td>
<td>0.125134</td>
<td>14914</td>
<td>54</td>
</tr>
<tr>
<td>64</td>
<td>0.124980</td>
<td>56793</td>
<td>122</td>
</tr>
<tr>
<td>128</td>
<td>0.124941</td>
<td>216327</td>
<td>887</td>
</tr>
</tbody>
</table>

Table 9: Numerical quenching times, numbers of iterations and CPU times (seconds) obtained with the implicit Euler method for $\varepsilon = 0$

<table>
<thead>
<tr>
<th>$I$</th>
<th>$t_n$</th>
<th>$n$</th>
<th>CPU time</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.125142</td>
<td>3887</td>
<td>2</td>
</tr>
<tr>
<td>32</td>
<td>0.125092</td>
<td>14864</td>
<td>11</td>
</tr>
<tr>
<td>64</td>
<td>0.124944</td>
<td>56778</td>
<td>133</td>
</tr>
<tr>
<td>128</td>
<td>0.125012</td>
<td>216314</td>
<td>98</td>
</tr>
</tbody>
</table>

Table 10: Numerical quenching times, numbers of iterations and CPU times (seconds) obtained with the implicit Euler method for $\varepsilon = \frac{1}{100000}$

<table>
<thead>
<tr>
<th>$I$</th>
<th>$t_n$</th>
<th>$n$</th>
<th>CPU time</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.125734</td>
<td>3903</td>
<td>3</td>
</tr>
<tr>
<td>32</td>
<td>0.125117</td>
<td>14914</td>
<td>14</td>
</tr>
<tr>
<td>64</td>
<td>0.124980</td>
<td>56792</td>
<td>118</td>
</tr>
<tr>
<td>128</td>
<td>0.125033</td>
<td>216324</td>
<td>274</td>
</tr>
</tbody>
</table>
Remark 6.1

When $\epsilon = 0$ and $p = 1$, we know that the quenching time of the continuous solution of (1)--(3) is equal to 0.125. We have also seen in Remark 3.3 that the quenching time of the semidiscrete solution is equal to 0.125. We observe from Tables 1--10 that when $\epsilon$ decays to zero, then the numerical quenching time of the discrete solution goes to 0.125. When one examines tables 1, 2, 3 and 4 one sees that an important perturbation on the potential and the initial datum has a meaningful impact on the numerical quenching time.

**Figure 1** — Evolution of the discrete solution, source $f(U) = (U_i^{(n)})^{-p}$, $\epsilon = 1/10000$, $\varphi_i = \frac{0.00 + \epsilon \sin(\pi x)}{2}$, $I = 16$ (implicit scheme).

**Figure 2** — Evolution of the discrete solution, source $f(U) = (U_i^{(n)})^{-p}$, $\epsilon = 1/10000$, $\varphi_i = \frac{0.00 + \epsilon \sin(\pi x)}{2}$, $I = 16$ (explicit scheme).

**Figure 3** — Evolution of the discrete solution, source $f(U) = (U_i^{(n)})^{-p}$, $\epsilon = 1/10000$, $\varphi_i = \frac{0.00 + \epsilon \sin(\pi x)}{2}$, $I = 32$ (implicit scheme).

**Figure 4** — Evolution of the discrete solution, source $f(U) = (U_i^{(n)})^{-p}$, $\epsilon = 1/10000$, $\varphi_i = \frac{0.00 + \epsilon \sin(\pi x)}{2}$, $I = 32$ (explicit scheme).
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