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Convergence Of Discrete Scheme For The Quenching Solution Of A Semilinear Parabolic PDE With A Singular Nonlinearity

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***Abstract*—This paper concerns the study of the numerical approximations for the following boundary value problem.:**

$$\begin{cases} u_t(x,t)=u_{xx}(x,t)-c(x)f(u(x,t)) & x\in(0,1),t\in(0,T), \\ u_x(0,t)=0, \quad u_x(1,t)=0 & t\in(0,T), \\ u(x,0)=u_0(x)>0, & x\in[0,1] \end{cases}$$

Where $f:(0,+\infty)\to(0,+\infty)$ **is** C^1 **convex nondecreasing function,** $\lim_{s\to0^+} f(s)=\infty, \int_0^\alpha \frac{ds}{f(s)},$ **for any positive real** α **. The initial datum** $u_0\in C^2([0,1]), u_0'(0)=0$ **and** $u_0'(1)=0,$ **The potential** $c\in C^1([0,1]), c(x)>0, x\in(0,1), c'(0)=0, c'(1)=0.$

We find some conditions under which the solution of a semidiscrete form of the above problem quenches a finite time and estimate its semidiscrete quenching time. We also the semidiscrete quenching time converges to the real one when the mesh size goes to zero. A similar study has been also investigated taking a discrete form of the above problem. Finally, we give some numerical experiments to illustrate our analysis.

***AMS subject classification* (2000): *35B40, 35B50, 35K60, 65M06*.**

Key-words and phrase —Semidiscretizations, Semilinear heat Equation, quenching, numerical quenching time, convergence, Full discretizations.

1. Introduction

Consider the following boundary value problem

$$\begin{cases} u_t(x,t)=u_{xx}(x,t)-c(x)f(u(x,t)), & x\in(0,1),t\in(0,T) \quad (1) \\ u_x(0,t)=0, \quad u_x(1,t)=0, & t\in(0,T) \quad (2) \\ u(x,0)=u_0(x)>0, & x\in[0,1] \quad (3) \end{cases}$$

Where $f:(0,\infty)\to(0,\infty)$ is C^1 non decreasing function, and C^1 convexe function, $\int_0^\alpha \frac{ds}{f(s)}<\infty$ for any positve real α . $\lim_{s\to0^+} f(s)=+\infty,$

$$c\in C^1([0,1]), c(x)>0, x\in(0,1), c'(0)=0, c'(1)=0, u_0'(0)=0,$$

and $u'(1)=0,$ the initial data $u_0\in C^2([0,1]),$ $u_0(x)>0, \ x\in(0,1),$

$$u_0''(x)-c(x)f(u_0(x))<0, \quad x\in(0,1) \quad (4)$$

$$u_0'(x)>0, x\in(0,1) \quad (5)$$

$$u_0'(0)=0, \quad u_0'(1)=0. \quad (6)$$

Here $[0,T]$ is the maximal time interval on which the solution u of (1)-(3) exist. The time T may be finite or infinite. When T is finite, then we say that the solution u of (1)-(3) develops a singularity in a finite time , namely, $\lim_{t\to T} U_{min}(t) = 0,$

Where $U_{min}(t) = min_{0\le x\le 1} U(x,t).$

In this last case, we say that the solution u of (1)-(3) quenche in a finite time and the time T is called the quenching of the solution u.

The theorical study of solution for semi linear heat equations which quench in finite time has been the subject of investigation of many authors see [2], [7], [11]- [13], [21], [22], [18], [30], and the reference of classical solution has been proved and this solution is unique. In addition, it is shown that if the initial data at (3) satisfies

$$u_0''(x) - c(x)u_0^{-q}(x) \le -Bu_0^{-q}(x), \quad x \in [0,1]$$

Where $B \in (0,1]$ and $q > 0$, then the classical solution u of (1)-(3) Quenches in afinite time T and we have the following estimate

$$\frac{\min_{0\le x\le 1}(u_0(x))^{q+1}}{q+1} \le T \le \frac{\min_{0\le x\le 1}(u_0(x))^{q+1}}{B(q+1)},$$

$$(B(q+1))^{\frac{1}{q+1}}(T-t)^{\frac{1}{q+1}} \le \min_{0\le x\le 1}(t) \le (q+1)^{\frac{1}{q+1}}(T-t)^{\frac{1}{q+1}}$$

(See, For in stance ([7], [11], [12])).

In this article, we are interested in the numerical study of the phenomenon of quenching. Our aim is to build a semidiscrete scheme where solution obeys the property of the continuous one. In order to facilitate our discussion, let us notice that the first condition in (4) allows the solution u to attain its minimum at the point $x = 0,$ and the second one permits the solution u to decrease with respect to the second variable. The hypotheses (5) are compatibility condition which ensure the regularity of the solution.

This paper is organized as follows. In the next section, we give some results about the discrete maximum principle. In the third section, under some conditions, we prove that the solution of a semidiscrete form of (1) --(3) quenches in a finite

time and estimate its semidiscrete quenching time. In the fourth section, we prove the convergence of the semidiscrete quenching time. In the fifth section, we study the results of sections 3 and 4 taking a discrete form of (1) --(3). Finally, in the last section, we give some numerical results to illustrate our analysis.

2- Properties of semi discrete problem

We start our study by the construction of a semidiscrete scheme as follows. Let I be a positive integer, and let $h = \frac{1}{I}.$ Define the grid $x_i = ih, 0 \le i \le I,$ and approximate the solution $U_i(0) = \varphi_i > 0, 0 \le i \le I,$ u of the problem (1)--(3) by the solution $U_h(t) = (U_0(t), U_1(t), \ldots, U_I(t))^T$ of the following semidiscrete equations

$$\frac{dU_i(t)}{dt} - \delta^2 U_i(t) = -\beta_i f(U_i(t)), 0 \le i \le I, t \in (0, T_q^h),$$
(7)

$$U_I(t) = 1, t \in (0, T_q^h), \quad (8)$$

$$U_i(0) = \varphi_i > 0, 0 \le i \le I, \quad (9)$$

Where

$$\delta^2 U_i(t) = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2}, 1 \le i \le I-1,$$

where $\beta_i > 0, \varphi_i > 0$

$$\delta^2 U_0(t) = \frac{2U_1(t) - 2U_0(t)}{h^2}, \quad \delta^2 U_I(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2} \quad . (10)$$

β_i and φ_i are approximations of $c(x_i)$ and $u_0(x_i),$ respectively. There is another motivation wich has cited our choice, one may remark our

sheme, we have not chosen $\beta_i = c(x_i)$ and $\varphi_i = u_0(x_i).$ The motivation to give the inital data and the potential of this manner is two fold. Firstly, in a lot of situations, it is difficult to have either the exact value of the potential or that of the initial data. It is the case when one of there is, for instance, the solution of a complicated ordianry differential equation. Secondly we want to study the behavior of the quenching time when one pertubs slightly either the potential or the initial datum.

Here $(0, T_q^h)$ is the maximal time interval on which

where $\| U_h(t) \|_{\inf} = \min_{0\le i\le I} U_i(t).$

When the time T_q^h is finite, we say that the solution $U_h(t)$ of (7)-(9) quenches in a finite time and the time T_q^h is called the quenching time of the solution $U_h(t).$

Definition 2.1

We say that the $U_h(t)$ solutionof (7)-(9) quenches in a finite time if there exist a finite time T_q^h such that $U_{min}(t) > 0,$ for $t \in (0, T_q^h)$ but $\lim_{t\to T_q^h} U_{hmin}(t) = 0,$ where $U_{hmin}(t) = min_{0\le x\le 1} U_i(t).$

The time T_q^h is called the quenching time of the solution $U_h(t)$.

The following lemma is a semidiscrete form of the maximum principle.

Lemma 2.1

Let $\alpha_h(t) \in C^0([0,T), \mathbb{R}^{I+1})$ and let

$V_h \in C^1([0,T), \mathbb{R}^{I+1})$ be such that

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + \alpha_i(t) V_i(t) \geq 0, 0 \leq i \leq I, t \in (0,T),$$

(11)

$V_i(0) \geq 0, 0 \leq i \leq I$. (12)

Then $V_i(t) \geq 0, 0 \leq i \leq I, t \in (0,T)$.

Proof:

Let T_0 be any quantity satisfying the inequality $T_0 < T$ and define the vector $Z_h(t) = e^{\lambda t} V_h(t)$ where λ is such that

$$\alpha_i(t) - \lambda > 0 \quad \text{for} \quad 0 \leq i \leq I, \quad t \in [0, T_0].$$

Set $m = \min_{0 \leq t \leq T_0} \| Z_h(t) \|_{\inf}$. Since $Z_h(t)$ is a continuous vector on the compact $[0, T_0]$, there exist $i_0 \in \{0,...,I\}$ and $t_0 \in [0, T_0]$ such that $m = Z_{i_0}(t_0)$. We observe that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0, \quad (13)$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \geq 0.$$

(14)

Since $Z_{i_0}(t_0) \leq Z_{i_0}(t_0 - k), Z_{i_0+1}(t_0) \geq Z_{i_0}(t_0)$, and $Z_{i_0-1}(t_0) \geq Z_{i_0}(t_0)$. From(11), we obtain the following inequality

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + (\alpha_{i_0}(t_0) - \lambda) Z_{i_0}(t_0) \geq 0.$$

(15)

We deduce from (13)-(15) that $(\alpha_{i_0}(t_0) - \lambda) Z_{i_0}(t_0) \geq 0$, which implies that $Z_{i_0}(t_0) \geq 0$. Therefore, $V_h(t) \geq 0$ for $t \in [0, T_0]$ and the proof is complete. □

Another form of the maximum principle for semidiscrete equations is the following comparison lemma.

Lemma 2.2

Let $f \in C^0(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. If

$V_h, W_h \in C^1([0,T), \mathbb{R}^{I+1})$ are such that

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + f(V_i(t), t) <$$
$$\frac{dW_i(t)}{dt} - \delta^2 W_i(t) + f(W_i(t), t), \quad 0 \leq i \leq I, t \in (0,T),$$

$V_i(0) < W_i(0), 0 \leq i \leq I,$ then $V_i(t) < W_i(t),$ $0 \leq i \leq I, \quad t \in (0,T).$

Proof:

Let $Z_h(t) = W_h(t) - V_h(t)$ and let t_0

be the first $t \in (0,T)$ such that $Z_h(t) > 0$ for $t \in [0, t_0)$ but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0,...,I\}$. We see that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0,$$

$\delta^2 Z_{i_0}(t_0) \geq 0$. Therefore, we have

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + f(W_{i_0}(t_0), t_0) - f(V_{i_0}(t_0), t_0) \leq 0,$$

which contradicts the first strict inequality of the lemma and this ends the proof.□

Lemma 2.3

Let U_h be the solution of (7)-(9). Assume that the initial data at (9) satisfies $\varphi_i < 1, 1 \leq i \leq I-1$. Then, we have $U_i(t) < 1, t \in (0, T_q^h)$.

Proof:

Let $t_0 \in (0, T_q^h)$ be the first time $t \in (0, T_q^h)$ such that $U_i(t) < 1$, for $1 \leq i \leq I-1, t \in (0, t_0)$ such that $U_i(t) < 1$, for $1 \leq i \leq I-1, t \in (0, t_0)$, but $U_j(t_0) = 1$ for certain $j \in \{1,...,I-1\}$. We have

$$\frac{dU_j(t_0)}{dt} = \lim_{k \to 0} \frac{U_j(t_0) - U_j(t_0 - k)}{k} \geq 0 \quad \text{and}$$

$$\delta^2 U_j(t_0) = \frac{U_{j+1}(t_0) - 2U_j(t_0) + U_{j-1}(t_0)}{h^2}.$$

Which implies that

$\dfrac{dU_j(t_0)}{dt} - \delta^2 U_j(t_0) + \beta_j f(U_j(t_0)) > 0.$ But, this contradicts (7) and the proof is complete.□

3- Semi discrete quenching solutions

In this section, under some assumptions,we show that the solution U_h of (7)--(9) quenches in a finite time and estimate its semidiscrete quenching time.

We need the following result about the operator δ^2.

Lemma 3.1

Let $U_h \in \mathbb{R}^{I+1}$ be such that $U_h > 0$. Then, we have

$$\delta^2(f(U))_i \geq f'(U_i)\delta^2 U_i, \quad 0 \leq i \leq I.$$

Proof:

Applying Taylor's expansion, we find that

$$\delta^2(f(U))_i = f'(U_i)\delta^2 U_i + \frac{(U_{i+1}-U_i)^2}{h^2} f''(\theta_i) + \frac{(U_{i-1}-U_i)^2}{h^2} f''(\eta_i), \quad 0 \leq i \leq I,$$

where θ_i is an intermediate value between U_i and U_{i+1}, η_i the one between U_{i-1} and $U_i, U_{-1} = U_1, U_{I+1} = U_{I-1}, \eta_0 = \theta_0$, $\eta_I = \theta_I$. Use the fact that $U_h > 0$ to complete the rest of the proof. □ The statement of the result about solutions which quench in a finite time is the following.

Theorem 3.1

Let U_h be the solution of (7)--(9) and assume that there exists a positive constant $A \in (0,1]$ and the initial data at (9) satisfies

$$\delta^2\varphi_i - \beta_i f(\varphi_i) \leq -Af(\varphi_i), 0 \leq i \leq I. \quad (16)$$

Then, the solution U_h quenches in a finite time T_q^h and we have the following estimate

$$T_q^h \leq \frac{1}{A}\int_0^{\|\varphi_h\|_{inf}} \frac{d\sigma}{f(\sigma)}.$$

Proof:

Since $(0, T_q^h)$ is the maximal time interval on which $\| U_h(t) \|_{\inf} > 0$, our aim is to show that T_q^h is finite and satisfies the above inequality.

Introduce the vector $J_h(t)$ defined as follows

$$J_i(t) = \frac{dU_i(t)}{dt} + Af(U_i(t)), \ 0 \leq i \leq I.$$

A straightforward calculation gives

$$\frac{dJ_i}{dt} - \delta^2 J_i = \frac{d}{dt}\left(\frac{dU_i}{dt} - \delta^2 U_i\right) + Af'(U_i(t))\frac{dU_i}{dt} - A\delta^2(f(U_i(t)))_i, 0 \leq i \leq I.$$

From Lemma 3.1, we have $\delta^2(f(U))_i \geq f'(U_i)\delta^2 U_i, 0 \leq i \leq I,$

which implies that

$$\frac{dJ_i}{dt} - \delta^2 J_i \leq \frac{d}{dt}\left(\frac{dU_i}{dt} - \delta^2 U_i\right) + Af'(U_i)\left(\frac{dU_i}{dt} - \delta^2 U_i\right), 0 \leq i \leq I.$$

Using (7), we arrive at

$$\frac{dJ_i}{dt} - \delta^2 J_i \leq -\beta_i f'(U_i) J_i, \quad 0 \leq i \leq I, \quad t \in (0, T_q^h).$$

From (16), we observe that $J_h(0) \leq 0$. We deduce from Lemma 2.1 that $J_h(t) \leq 0$ for $t \in (0, T_q^h)$, which implies that

$$\frac{dU_i(t)}{dt} \leq -Af(U_i(t)), \quad 0 \leq i \leq I, \quad t \in (0, T_q^h).$$

These estimates may be rewritten in the following form $\frac{dU_i}{f(U_i)} \leq -Adt, 0 \leq i \leq I$. Integrating the above inequalities over the interval (t, T_q^h), we get

$$T_q^h - t \leq \frac{1}{A}\int_0^{U_i(t)} \frac{d\sigma}{f(\sigma)}, \ 0 \leq i \leq I.$$

Using the fact that $\| \varphi_h \|_{\inf} = U_{i_0}(0)$ for a certain $i_0 \in \{0, ..., I\}$ and taking $t = 0$ in (18), we obtain the desired result. □

Remark 3.1

The inequalities (18) imply that

$$T_q^h - t_0 \leq \frac{1}{A}\int_0^{\|U_h(t_0)\|_{\inf}} \frac{d\sigma}{f(\sigma)} \quad for \quad t_0 \in (0, T_q^h),$$

and

$\| U_h(t) \|_{\inf} \geq H(A(T_q^h - t)) \quad for \quad t \in (0, T_q^h),$

where $H(s)$ is the inverse of the function

$$F(s) = \int_0^s \frac{d\sigma}{f(\sigma)}.$$

Remark 3.2

Let U_h be the solution of (7)-(9). Then, we have

$$T_q^h \geq \frac{1}{\| \beta_h \|_\infty} \int_0^{\|\varphi_h\|} \frac{d\sigma}{f(\sigma)},$$ and

$$\| U_h(t) \|_{\inf} \leq \| \beta_h \|_\infty H(A(T_q^h - t)) \quad for \quad t \in (0, T_q^h).$$

To prove these estimates, we proceed as follows. Introduce the function $v(t)$ defined as follows $v(t) = \| U_h(t) \|_{\inf}$ for $t \in [0, T_q^h)$. Let $t_1, t_2 \in [0, T_q^h)$. Then, there exist $i_1, i_2 \in \{0, ..., I\}$ such that $v(t_1) = U_{i_1}(t_1)$ and $v(t_2) = U_{i_2}(t_2)$. We observe that

$$v(t_2) - v(t_1) \geq U_{i_2}(t_2) - U_{i_2}(t_1) = (t_2 - t_1)\frac{dU_{i_2}(t_2)}{dt} + o(t_2 - t_1),$$

$$v(t_2) - v(t_1) \leq U_{i_1}(t_2) - U_{i_1}(t_1) = (t_2 - t_1)\frac{dU_{i_1}(t_1)}{dt} + o(t_2 - t_1),$$

which implies that $v(t)$ is Lipschitz continuous. Further, if $t_2 > t_1$, then

$$\frac{v(t_2) - v(t_1)}{t_2 - t_1} \geq \frac{dU_{i_2}(t_2)}{dt} + o(1) = \delta^2 U_{i_2}(t_2) - \beta_{i_2} f(U_{i_2}(t_2)) + o(1).$$

Obviously,

$\delta^2 U_{i_2}(t_2) \geq 0$. Letting $t_1 \to t_2$, and using the fact that $\beta_{i_2} \leq \| B_h \|_\infty$, we obtain $\frac{dv(t)}{dt} \geq -\| \beta_h \|_\infty f(v(t))$ for $t \in (0, T_q^h)$

Or equivalently $\frac{dv}{f(v(t))} \geq -\| \beta_h \|_\infty dt$ for $t \in (0, T_q^h)$. Integrate the above inequality over (t, T_q^h) to obtain $T_q^h - t \geq \frac{1}{\| \beta_h \|_\infty} \int_0^{v(t)} \frac{d\sigma}{f(\sigma)}$.

Since $v(t) = \| U_h(t) \|_{\inf}$, we arrive at

$$T_q^h - t \geq \frac{1}{\| \beta_h \|_\infty} \int_0^{\|U_h(t)\|_{\inf}} \frac{d\sigma}{f(\sigma)}$$ and

the second estimate follows. To obtain the first one, it suffices to replace t by 0 in the above inequality and use the fact that $\| \varphi_h \|_{\inf} = \| U_h(0) \|_{\inf}$.

Remark 3.3

If $\varphi_i = \alpha, 0 \leq i \leq I$, where α is a positive constant, then one may take $A = 1$. It may imply that the potential equals to 1. In this case,

$$T_q^h = \frac{\alpha^{p+1}}{p+1} \quad and \quad \| U_h(t) \|_{\inf} = (p+1)^{\frac{1}{p+1}} (T_q^h - t)^{\frac{1}{p+1}}$$

$$for \quad t \in (0, T_q^h).$$

4- Convergence of the semidiscrete quenching time

In this section, under some assumptions, we show that the solution of the semidiscrete problem quenches in a finite time and its semidiscrete quenching time converges to the real one when the mesh size goes to zero. We denote

$$c_h = (c(x_0), ..., c(x_I))^T, u_h(t) = (u(x_0, t), ..., u(x_I, t))^T$$

and $\quad \| U_h(t) \|_\infty = \max_{0 \leq i \leq I} | U_i(t) |.$

In order to obtain the convergence of the semidiscrete quenching time, we firstly prove the following theorem about the convergence of the semidiscrete scheme.

Theorem 4.1

Assume that the problem (1)-(3) has a solution $u \in C^{4,1}([0,1] \times [0, T])$ such that

$\min_{t \in [0,T]} u_{\min}(t) = \tilde{n} > 0$. Suppose that the potential at (7) and the initial data at (9) satisfy

$$\| \varphi_h - u_h(0) \|_\infty = o(1) \quad as \quad h \to 0, \quad (19)$$

$$\| \beta_h - c_h \|_\infty = o(1) \quad as \quad h \to 0. \quad (20)$$

Then, for h sufficiently small, the problem (7)-(9) has a unique solution $U_h \in C^1([0,T], \mathbb{R}^{I+1})$ such that the following relation holds

$$\max_{0\le t\le T} \| \beta_h - c_h \|_\infty = 0(\| \varphi_h - u_h(0) \|_\infty + h^2) \quad as \quad h \to 0.$$

Proof:

Let $K > 0$ and $L > 0$ be such that

$$\frac{\| u_{xxxx} \|_\infty}{12} \le K, \quad f(\frac{\rho}{2}) \le K \quad \text{and} \qquad (21)$$
$$-(\| c_h \|_\infty + 1) f'(\frac{\rho}{2}) \le L.$$

The problem (7)-(9) has for each h, a unique solution $U_h \in C^1([0,T_q^h), \mathbb{R}^{I+1})$. Let $t(h) \le \min\{T, T_q^h\}$ be the greatest value of $t > 0$ such that

$\| U_h(t) - u_h(t) \|_\infty < \frac{\tilde{n}}{2}$ for $t \in (0, t(h))$. (22)

The relation (19) implies that $t(h) > 0$ for h sufficiently small. By the triangle inequality, we obtain

$\| U_h(t) \|_{\inf} \ge \| u_h(t) \|_{\inf} - \| U_h(t) - u_h(t) \|_\infty$ for $t \in (0, t(h))$,which implies that

$\| U_h(t) \|_{\inf} \ge \tilde{n} - \frac{\tilde{n}}{2} = \frac{\tilde{n}}{2}$ for $t \in (0, t(h))$. (23)

Since $u \in C^{4,1}$, taking the derivative in x on both sides of (1) and due to the fact that u_x, u_{xt} vanish at $x = 0$ and $x = 1$, we observe that u_{xxx} also vanishes at $x = 0$ and

$x = 1$. Applying Taylor's expansion, we discover that

$$u_{xx}(x_i,t) = \delta^2 u(x_i,t) - \frac{h^2}{12} u_{xxxx}(x_i,t),$$
$$0 \le i \le I, \quad t \in (0, t(h)).$$

To establish the above equalities for $i = 0$ and $i = I$, we have used the fact that u_x and u_{xxx} vanish at $x = 0$ and $x = 1$. A direct calculation yields

$$u(x_i,t) - \delta^2 u(x_i,t) = -\beta_i f(u(x_i,t)) - \frac{h^2}{12} u_{xxxx}(x_i,t)$$
$$+(\beta_i - c(x_i)) f(u(x_i,t)), \quad 1 \le i \le I-1.$$

Let $e_h(t) = U_h(t) - u_h(t)$ be the error of discretization. From the mean value theorem, we have

$$\frac{de_i(t)}{dt} - \delta^2 e_i(t) = -\beta_i f'(\theta_i) e_i + \frac{h^2}{12} u_{xxxx}(x_i,t)$$
$$-(\beta_i - c(x_i)) f(u(x_i,t)), 0 \le i \le I, t \in (0, t(h)),$$

where θ_i is an intermediate value between

$U_i(t)$ and $u(x_i,t)$. Using (21), (22), we arrive at

$$\frac{de_i(t)}{dt} - \delta^2 e_i(t) \le L | e_i(t) | + Kh^2 +$$
$$K \| \beta_h - c_h \|_\infty, 0 \le i \le I, \quad t \in (0, t(h)).$$

Introduce the vector $z_h(t)$ defined as follows

$$z_i(t) = e^{(L+1)t} (\| \varphi_h - u_h(0) \|_\infty + Kh^2 +$$
$$K \| \beta_h - c_h \|_\infty), \quad 0 \le i \le I, t \in (0, t(h)).$$

A straightforward computation reveals that

$$\frac{dz_i}{dt} - \delta^2 z_i > L | z_i | + Kh^2 + K \| \beta_h - c_h \|_\infty,$$
$$0 \le i \le I, \quad t \in (0, t(h)),$$

$$z_i(0) > e_i(0), 0 \le i \le I.$$

It follows from Comparison Lemma 2.2 that

$z_i(t) > e_i(t)$ for $t \in (0, t(h))$, $\quad 0 \le i \le I.$

In the same way, we also prove that

$z_i(t) > -e_i(t)$ for $t \in (0, t(h))$, $\quad 0 \le i \le I$,

which implies that

$\| U_h(t) - u_h(t) \|_\infty \le e^{(L+1)t} (\| \varphi_h - u_h(0) \|_\infty + Kh^2 + K \| \beta_h - c_h \|_\infty)$ for $t \in (0, t(h))$.

Let us show that $t(h) = \min\{T, T_q^h\}$. Suppose that

$t(h) < \min\{T, T_q^h\}$. From (22), we obtain

$$\frac{\tilde{n}}{2} \le \| U_h(t(h)) - u_h(t(h)) \|_\infty \le e^{(L+1)T} (\| \varphi_h - u_h(0) \|_\infty +$$
$$Kh^2 + K \| \beta_h - c_h \|_\infty).$$

Let us notice that both last formulas for $t(h)$ are valid for sufficiently small h. Since the term on the right hand side of the above inequality goes to zero as h goes to zero, we deduce that $\frac{\tilde{n}}{2} \le 0$, which is impossible. Consequently $t(h)=\min\{T,T_q^h\}$. Now, let us show that $t(h)=T$.

Suppose that $t(h)=T_q^h<T$. Reasoning as above, we prove that

we have a contradiction and the proof is complete

Now, we are in a position to prove the main theorem of this section.

Theorem 4.2

Suppose that the problem (1)-(3) has a solution u which quenches in a finite time T_q such that $u\in C^{4,1}([0,1]\times[0,T_q))$. Assume that the potential at (7)and the initial data at (9) satisfy the conditions (19) and (20), respectively. Under the hypothesis of Theorem 3.1, the problem (7)-(9) has a solution U_h which quenches in a finite time

T_q^h and we have $\lim_{h\to0}T_q^h=T_q$.

Proof:

Let $0<\varepsilon<T_q/2$. There exists $ñ\in(0,1)$ such that

$$\frac{1}{A}\int_0^{ñ}\frac{d\sigma}{f(\sigma)}\le\frac{\varepsilon}{2}. \quad (24)$$

Since u quenches in a finite time T_q, there exist

$h_0(\varepsilon)>0$ and a time $T_0\in(T_q-\frac{\varepsilon}{2},T_q)$ such that

$0<u_{\min}(t)<\frac{ñ}{2}$ for $t\in[T_0,T_q)$, $h\le h_0(\varepsilon)$. It is not hard to see that

$u_{\min}(t)>0$ for $t\in[0,T_0]$, $h\le h_0(\varepsilon)$.

From Theorem 4.1, the problem (7)-(9) has a solution $U_h(t)$ and we get $\| U_h(t)-u_h(t)\|_\infty\le\frac{ñ}{2}$

for $t\in[0,T_0]$, $h\le h_0(\varepsilon)$, which implies that

$\| U_h(T_0)-u_h(T_0)\|_\infty\le\frac{ñ}{2}$ for $h\le h_0(\varepsilon)$. Applying the triangle inequality, we find that

$$\| U_h(T_0)\|_{\inf}\le\| U_h(T_0)-u_h(T_0)\|_\infty+\| u_h(T_0)\|_{\inf}\le\frac{ñ}{2}+\frac{ñ}{2}=ñ$$

for $h\le h_0(\varepsilon)$.

From Theorem 3.1, $U_h(t)$ quenches at the time T_q^h . We deduce from Remark 3.1 and (22) that for $h\le h_0(\varepsilon)$

$$|T_q^h-T_q|\le|T_q^h-T_0|+|T_0-T_q|\le\frac{1}{A}\int_0^{\|U_h(T_0)\|_{\inf}}\frac{d\sigma}{f(\sigma)}+\frac{\varepsilon}{2}\le\varepsilon$$

which leads us to the desired result. □

5- Full discretizations

In this section, we study the phenomenon of quenching using a full discrete explicit scheme of (1)-(3). Approximate the solution $u(x,t)$ of the problem (1)-(3) by the solution $U_h^{(n)}=(U_0^{(n)},U_1^{(n)},\dots,U_I^{(n)})^T$ of the following explicit scheme

$$\delta_t U_i^{(n)}=\delta^2U_i^{(n)}-\beta_i f(U_i^{(n)}),\ 0\le i\le I, \quad (25)$$

$$U_i^{(0)}=\varphi_i>0,\ \ 0\le i\le I, \quad (26)$$

where $n\ge0$,

$$\delta_t U_i^{(n)}=\frac{U_i^{(n+1)}-U_i^{(n)}}{\Delta t_n}.$$

$$\left(\frac{f(s)}{s}\right)'=\frac{f'(s)s-f(s)}{s^2}\le0,\quad \text{for}\quad s>0.$$

$U_h^{(n)}>0$, then $-\frac{f(U_i^{(n)})}{U_i^{(n)}}\ge-\frac{f(\| U_h^{(n)}\|_{\inf})}{\| U_h^{(n)}\|_{\inf}}$,

$0\le i\le I$, and a straightforward computation reveals that

$$U_0^{(n+1)}\ge\frac{2\Delta t_n}{h^2}U_1^{(n)}+\left(1-2\frac{\Delta t_n}{h^2}-\|\beta_h\|_\infty\Delta t_n\frac{f(\| U_h^{(n)}\|_{\inf})}{\| U_h^{(n)}\|_{\inf}}\right)U_0^{(n)}$$

$$U_i^{(n+1)}\ge\frac{\Delta t_n}{h^2}U_{i+1}^{(n)}+\left(1-2\frac{\Delta t_n}{h^2}-\|\beta_h\|_\infty\Delta t_n\frac{f(\| U_h^{(n)}\|_{\inf})}{\| U_h^{(n)}\|_{\inf}}\right)U_i^{(n)}$$

$$+\frac{\Delta t_n}{h^2}U_{i-1}^{(n)},\ 1\le i\le I-1,$$

$$U_I^{(n+1)}\ge\frac{2\Delta t_n}{h^2}U_{I-1}^{(n)}+\left(1-2\frac{\Delta t_n}{h^2}-\|\beta_h\|_\infty\Delta t_n\frac{f(\| U_h^{(n)}\|_{\inf})}{\| U_h^{(n)}\|_{\inf}}\right)U_I^{(n)}.$$

In order to permit the discrete solution to reproduce the properties of the continuous one when the time t approaches the quenching time T_q, we need to adapt the size of the time step so that we choose

$$\Delta t_n = \min\{\frac{(1-\tau)h^2}{2}, \tau\frac{f(\| U_h^{(n)} \|_{\inf})}{\| U_h^{(n)} \|_{\inf}}\}$$

With $0<\tau<1$. We observe that $1-2\frac{\Delta t_n}{h^2} - \| \beta_h \|_\infty \Delta t_n \frac{f(\| U_h^{(n)} \|_{\inf})}{\| U_h^{(n)} \|_{\inf}} \geq 0,$

Which implies that $U_h^{(n+1)} > 0$. Thus, since by hypothesis $U_h^{(0)} = \varphi_h > 0$, if we take Δt_n as defined above, then using a recursion argument, we see that the positivity of the discrete solution is guaranteed. Here, τ is a parameter which will be chosen later to allow the discrete solution $U_h^{(n)}$ to satisfy certain properties useful to get the convergence of the numerical quenching time defined below. If necessary, we may take

$$\Delta t_n = \min\{\frac{(1-\tau)h^2}{K}, \tau\frac{f(\| U_h^{(n)} \|_{\inf})}{\| U_h^{(n)} \|_{\inf}}\}$$

with $K>2$ because in this case, the positivity of the discrete solution is also guaranteed. The following lemma is a discrete form of the maximum principle.

Lemma 5.1

Let $a_h^{(n)}$ and $V_h^{(n)}$ be two sequences such that $a_h^{(n)}$ is bounded and

$$\delta_t V_i^{(n)} - \delta^2 V_i^{(n)} + a_i^{(n)} V_i^{(n)} \geq 0,$$

$0 \leq i \leq I,\ n \geq 0,$ (27)

$V_i^{(0)} \geq 0,\ 0 \leq i \leq I.$ (28)

Then $V_i^{(n)} \geq 0$ for $n \geq 0,$ \; $0 \leq i \leq I$

if $\Delta t_n \leq \frac{h^2}{2 + \| a_h^{(n)} \|_\infty h^2}.$

Proof:

If $V_h^{(n)} \geq 0$, then a routine computation yields

$$V_0^{(n+1)} \geq \frac{2\Delta t_n}{h^2} V_1^{(n)} + (1 - 2\frac{\Delta t_n}{h^2} - \Delta t_n \| a_h^{(n)} \|_\infty) V_0^{(n)},$$

$$V_i^{(n+1)} \geq \frac{\Delta t_n}{h^2} V_{i+1}^{(n)} + (1 - 2\frac{\Delta t_n}{h^2} - \Delta t_n \| a_h^{(n)} \|_\infty) V_i^{(n)} + \frac{\Delta t_n}{h^2} V_{i-1}^{(n)},$$

$1 \leq i \leq I-1,$

$$V_I^{(n+1)} \geq \frac{2\Delta t_n}{h^2} V_{I-1}^{(n)} + (1 - 2\frac{\Delta t_n}{h^2} - \Delta t_n \| a_h^{(n)} \|_\infty) V_I^{(n)}.$$

Since $\Delta t_n \leq \frac{h^2}{2 + \| a_h^{(n)} \|_\infty h^2}$,

we see that $t_n \| a_h^{(n)} \|_\infty$

is nonnegative. From (27), we deduce by induction that $V_h^{(n)} \geq 0$ which ends the proof. □

A direct consequence of the above result is the following

comparison lemma. Its proof is straightforward.

Lemma 5.2

Let $V_h^{(n)}, W_h^{(n)}$ and $a_h^{(n)}$ be three sequences

such that $a_h^{(n)}$ is bounded and

$$\delta_t V_i^{(n)} - \delta^2 V_i^{(n)} + a_i^{(n)} V_i^{(n)} \leq \delta_t W_i^{(n)} - \delta^2 W_i^{(n)} + a_i^{(n)} W_i^{(n)},$$

$0 \leq i \leq I,\quad n \geq 0,$

$V_i^{(0)} \leq W_i^{(0)},\ 0 \leq i \leq I.$

Then $V_i^{(n)} \leq W_i^{(n)}$ for $n \geq 0,\quad 0 \leq i \leq I$ if

$$\Delta t_n \leq \frac{h^2}{2 + \| a_h^{(n)} \|_\infty h^2}.$$

Now, let us give a property of the operator δ_t stated in the following lemma. Its proof is quite similar to that of Lemma 3.1, so we omit it here.

Lemma 5.3

Let $U^{(n)} \in \square$ be such that $U^{(n)} > 0$ for $n \geq 0$. Then, we have

$$\delta_t f(U^{(n)}) \geq f'(U^{(n)}) \delta_t U^{(n)},\quad n \geq 0.$$

Lemma 5.4

Let a, b be tow positive numbers such that $b<1$. then following estimate holds

$$\sum_{n=0}^{\infty} \frac{ab^n}{f(ab^n)} \leq \frac{a}{f(a)} - \frac{1}{\ln(b)} \int_0^a \frac{d\sigma}{f(\sigma)}.$$

Proof: We have

$$\int_0^\infty \frac{ab^x}{f(ab^x)} = \sum_{n=0}^{\infty} \int_n^{n+1} \frac{ab^x dx}{f(ab^x)}.$$

We observe that $ab^x \geq ab^{n+1}$ for $n \leq x \leq n+1,$

which that $\int_n^{n+1} \frac{ab^x dx}{f(ab^x)} \geq \frac{ab^{n+1}}{f(ab^{n+1})}.$

Consequently, we get

$$\int_n^{n+1} \frac{ab^x dx}{f(ab^x)} \geq \sum_{n=0}^{\infty} \int_n^{n+1} \frac{ab^x dx}{f(ab^x)} = -\frac{a}{f(a)} + \sum_{n=0}^{\infty} \frac{ab^n}{f(ab^n)}$$

.

Use the fact that

$$\int_0^{\infty} \frac{ab^x}{f(ab^x)} = -\frac{1}{\ln(b)} \int_0^a \frac{d\sigma}{f(\sigma)}$$

to complete the rest of the proof.□

The theorem below is the discrete version of Theorem 4.1.

Theorem 5.1

Suppose that the problem (1)--(3) has a solution $u \in C^{4,2}([0,1]\times[0,T])$ such that

$\min_{t\in[0,T]} u_{\min}(t) = \rho > 0$. Assume that the initial data at (26) satisfies the condition (16). Then, the problem (25)--(26) has a solution $U_h^{(n)}$ for h sufficiently small, $0 \leq n \leq J$ and the following relation holds

$$\max_{0\leq n\leq J} \| U_h^{(n)} - u_h(t_n) \|_\infty = O(\| \varphi_h - u_h(0) \|_\infty + \| c_h - \beta_h \|_\infty + h^2) \text{ as } h \to 0.$$

where J is any quantity satisfying the inequality $\sum_{n=0}^{J-1} \Delta t_n \leq T$ and $t_n = \sum_{j=0}^{n-1} \Delta t_j$.

Proof:

For each h, the problem (25)-(26) has a solution $U_h^{(n)}$. Let $N \leq J$ be the greatest value of n such that

$$\| U_h^{(n)} - u_h(t_n) \|_\infty < \frac{\rho}{2} \quad \text{for} \quad n < N. \quad (29)$$

We know that $N \geq 1$ because of (16). Applying the triangle inequality, we have

$$\| U_h^{(n)} \|_{\inf} \geq \| u_h(t_n) \|_{\inf} - \| U_h^{(n)} - u_h(t_n) \|_\infty \geq \frac{\rho}{2} \quad \text{for} \quad n < N. \quad (30)$$

As in the proof of Theorem 4.1, using Taylor's expansion, we find

that for $n < N$, $0 \leq i \leq I$,

$$\delta_t u(x_i,t_n) - \delta^2 u(x_i,t_n) + \beta_i f(u(x_i,t_n)) + (c(x_i) - \beta_i) f(u(x_i,t_n)) = -\frac{h^2}{12} u_{xxxx}(x_i,t_n) + \frac{\Delta t_n}{2} u_{tt}(x_i,t_n).$$

Let $e_h^{(n)} = U_h^{(n)} - u_h(t_n)$ be the error of discretization. From the mean value theorem, we get for $n < N, \ 0 \leq i \leq I$,

$$\delta_t e_i^{(n)} - \delta^2 e_i^{(n)} = -\beta_i f'(\xi_i^{(n)}) e_i^{(n)} + \frac{h^2}{12} u_{xxxx}(x_i,t_n) - \frac{\Delta t_n}{2} u_{tt}(x_i,t_n) + (c(x_i) - \beta_i) f(u(x_i,t_n)),$$

where $\xi_i^{(n)}$ is an intermediate value between $u(x_i,t_n)$ and $U_i^{(n)}$. Since $u_{xxxx}(x,t)$, $u_{tt}(x,t)$ are bounded, $u(x,t) \geq \rho$ and $\Delta t_n = O(h^2)$, then there exists a positive constant M such that

$$\delta_t e_i^{(n)} - \delta^2 e_i^{(n)} \leq -\beta_i f'(\xi_i^{(n)}) e_i^{(n)} + M \| c_h - \beta_h \|_\infty + Mh^2, \quad 0 \leq i \leq I, n < N.$$

Set $L = -(\| c_h \| \infty + 1) f'(\frac{\rho}{2})$ and introduce the vector $V_h^{(n)}$ defined as follows

$$V_i^{(n)} = e^{(L+1)t_n} (\| \varphi_h - u_h(0) \|_\infty + Mh^2 + M \| c_h - \beta_h \|_\infty), \quad 0 \leq i \leq I, n < N.$$

A straightforward computation gives

$$\delta_t V_i^{(n)} - \delta^2 V_i^{(n)} > -\beta_i f'(\xi_i^{(n)}) V_i^{(n)} + Mh^2 + M \| c_h - \beta_h \|_\infty, \quad 0 \leq i \leq I, n < N,$$

$$V_i^{(0)} > e_i^{(0)}, 0 \leq i \leq I.$$

We observe from (29) that $-\beta_i f'(\xi_i^{(n)})$ is bounded

from above by L. It follows from Comparison Lemma 5.2 that $V_h^{(n)} \geq e_h^{(n)}$. By the same way, we also prove that $V_h^{(n)} \geq -e_h^{(n)}$, which implies that

$$\| U_h^{(n)} - u_h(t_n) \|_\infty \leq e^{(L+1)t_n} (\| \varphi_h - u_h(0) \|_\infty + Mh^2 + M \| c_h - \beta_h \|_\infty), n < N.$$

Let us show that $N = J$. Suppose that $N < J$. If we replace n by N in (29) and use (30), we find that

$$\frac{\rho}{2} \leq \| U_h^{(N)} - u_h(t_N) \|_\infty \leq e^{(L+1)T} (\| \varphi_h - u_h(0) \|_\infty + Mh^2 + M \| c_h - \beta_h \|_\infty).$$

Since the term on the right hand side of the second inequality goes to zero as h goes to zero, we deduce that $\frac{\rho}{2} \leq 0$, which is a contradiction and the proof is complete. □ To handle the phenomenon of quenching for discrete equations, we need the following definition.

Definition 5.1

We say that the solution $U_h^{(n)}$ of 25)-(26) quenches in a finite time if\; $\| U_h^{(n)} \|_{\inf} > 0$ for $n \geq 0$, but

$$\lim_{n \to +\infty} \| U_h^{(n)} \|_{\inf} = 0 \quad and \quad T_h^{\Delta t} = \lim_{n \to \infty} \sum_{i=0}^{n-1} \Delta t_i < \infty.$$

The number $T_h^{\Delta t}$ is called the numerical quenching time of $U_h^{(n)}$. The following theorem reveals that the discrete solution

$U_h^{(n)}$ of (25)-(26) quenches in a finite time under some hypotheses.

Theorem 5.2

Let $U_h^{(n)}$ be the solution of (25)-(26). Suppose that there exists a constant $A \in (0,1]$ such that the initial data at (26) satisfies

$$\delta^2 \varphi_i - \beta_i f(\varphi_i) \leq -Af(\varphi_i), \qquad 0 \leq i \leq I. \quad (31)$$

Then $U_h^{(n)}$ is <u>nonincreasing</u> and quenches in a finite time $T_h^{\Delta t}$ which satisfies the following estimate

$$T_h^{\Delta t} \leq \frac{\tau \| \varphi_h \|_{\inf}}{f(\| \varphi_h \|_{\inf})} - \frac{\tau}{\ln(1-\tau')} \int_0^{\|\varphi_h\|_{\inf}} \frac{d\sigma}{f(\sigma)},$$

where

$$\tau' = A \min\{ \frac{(1-\tau)h^2 f(\| \varphi_h \|_{\inf})}{2 \| \varphi_h \|_{\inf}}, \tau \}.$$

Proof:

Introduce the vector $J_h^{(n)}$ defined as follows

$$J_i^{(n)} = \delta_t U_i^{(n)} + Af(U_i^{(n)}) \ , \quad 0 \leq i \leq I, \quad n \geq 0.$$

A straightforward computation yields for $0 \leq i \leq I, n \geq 0,$

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} = \delta_t \left(\delta_t U_i^{(n)} - \delta^2 U_i^{(n)} \right) + A\delta_t f(U_i^{(n)}) - A\delta^2 f(U_i^{(n)}).$$

Using (25), we arrive at

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} = -(\beta_i - A)\delta_t f(U_i^{(n)}) - A\delta^2 f(U_i^{(n)}),$$
$$0 \leq i \leq I, \quad n \geq 0.$$

It follows from Lemmas 5.3 and 3.1 that for $0 \leq i \leq I, n \geq 0,$

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} \leq -(\beta_i - A) f'(U_i^{(n)}) \delta_t U_i^{(n)} - Af'(U_i^{(n)}) \delta^2 U_i^{(n)}.$$

We deduce from (25) that

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} \leq -\beta_i f'(U_i^{(n)}) J_i^{(n)},$$
$$0 \leq i \leq I, n \geq 0.$$

Obviously, the inequalities (31) ensure that $J_h^{(0)} \leq 0$. Applying Lemma 5.1, we get $J_h^{(n)} \leq 0$, for $n \geq 0$, which implies that

$$U_i^{(n+1)} \leq U_i^{(n)} (1 - A\Delta t_n \frac{f((U_i^{(n)}))}{U_i^{(n)}}), \qquad (32)$$
$$0 \leq i \leq I, \quad n \geq 0.$$

These estimates reveal that the sequence $U_h^{(n)}$ is

<u>nonincreasing</u>. By induction, we obtain $U_h^{(n)} \leq U_h^{(0)} = \varphi_h$. Thus, the following holds

$$A\Delta t_n \frac{f(\| U_h^{(n)} \|_{\inf})}{\| U_h^{(n)} \|_{\inf}} \geq A \min\{ \frac{(1-\tau)h^2 f(\| \varphi_h \|_{\inf})}{2 \| \varphi_h \|_{\inf}}, \tau \} = \tau'. \qquad (33)$$

Let i_0 be such that $\| U_h^{(n)} \|_{\inf} = U_{i_0}^{(n)}$. Replacing i by i_0 in (38), we obtain

$$\| U_h^{(n+1)} \|_{\inf} \leq \| U_h^{(n)} \|_{\inf} (1-\tau'), \quad n \geq 0, \qquad (34)$$

and by iteration, we arrive at

$$\| U_h^{(n)} \|_{\inf} \leq \| U_h^{(0)} \|_{\inf} (1-\tau')^n = \| \varphi_h \|_{\inf} (1-\tau')^n, \quad n \geq 0. \qquad (35)$$

fSince the term on the right hand side of the above equality goes to zero as n approaches infinity, we conclude that $\| U_h^{(n)} \|_{\inf}$ tends to zero as n approaches infinity. Now, let us estimate the numerical quenching time. Due to (33) and the restriction $\Delta t_n \leq \frac{\tau \| U_h^{(n)} \|_{\inf}}{f(\| U_h^{(n)} \|_{\inf})}$,

it is not hard to see that

$$\Sigma_{n=0}^{+\infty} \Delta t_n \leq \tau \Sigma_{n=0}^{+\infty} \frac{\| \varphi_h \|_{\inf} (1-\tau')^n}{f(\| \varphi_h \|_{\inf} (1-\tau')^n)},$$

because $\frac{s}{f(s)}$ is nondecreasing for $s>0$. It follows from Lemma 5.4 that

$$\Sigma_{n=0}^{+\infty} \Delta t_n \leq \frac{\tau \| \varphi_h \|_{\inf}}{f(\| \varphi_h \|_{\inf})} - \frac{\tau}{\ln(1-\tau')} \int_0^{\| \varphi_h \|_{\inf}} \frac{d\sigma}{f(\sigma)}.$$

Use the fact that the quantity on the right hand side of the above inequality converges towards is finite to complete the rest of the proof. □

Remark 5.1 From (35), we deduce by induction that

$$\| U_h^{(n)} \|_{\inf} \leq \| U_h^{(q)} \|_{\inf} (1-\tau')^{n-q} \quad \text{for} \quad n \geq q,$$

and we see that

$$T_h^{\Delta t} - t_q = \Sigma_{n=q}^{+\infty} \Delta t_n \leq \tau \Sigma_{n=q}^{\infty} \frac{\| U_h^{(q)} \|_{\inf} (1-\tau')^{n-q}}{f(\| U_h^{(q)} \|_{\inf} (1-\tau')^{n-q})},$$

because $\frac{s}{f(s)}$ is nondecreasing for $s>0$. It follows from Lemma 5.4 that

$$T_h^{\Delta t} - t_q \leq \frac{\tau \| U_h^{(q)} \|_{\inf}}{f(\| U_h^{(q)} \|_{\inf})} - \frac{\tau}{\ln(1-\tau')} \int_0^{\| U_h^{(q)} \|_{\inf}} \frac{d\sigma}{f(\sigma)}.$$

Since

$\tau' = A\min\{\frac{(1-\tau)h^2 f(\| \varphi_h \|_{\inf})}{2 \| \varphi_h \|_{\inf}}, \tau\}$, if we take $\tau = h^2$, we get

$$\frac{\tau'}{\tau} = A\min\{\frac{(1-h^2)h^2 f(\| \varphi_h \|_{\inf})}{2 \| \varphi_h \|_{\inf}}, 1\} \geq A\min\{\frac{f(\| \varphi_h \|_{\inf})}{4 \| \varphi_h \|_{\inf}}, 1\}.$$

Therefore, there exist constants c_0, c_1 such that $0 \leq c_0 \leq \tau / \tau' \leq c_1$ and

$\frac{-\tau}{\ln(1-\tau')} = O(1)$, for the choice $\tau = h^2$.

In the sequel, we take $\tau = h^2$. Now, we are in a position to state the main theorem of this section.

Theorem 5.3

Suppose that the problem (1)-(3) has a solution u which quenches in a finite time T_q and $u \in C^{4,2}([0,1] \times [0, T_q))$. Assume that the initial data at (25) satisfies the condition (16). Under the

assumption of Theorem 5.2, the problem (25)-(26) has a solution $U_h^{(n)}$ which quenches in a finite time $T_h^{\Delta t}$ and the following relation holds

$$\lim_{h \to 0} T_h^{\Delta t} = T_q.$$

Proof.

We know from Remark 5.1 that $\frac{\tau}{\ln(1-\tau')}$ is bounded. Letting $0 < \varepsilon < T_q / 2$, there exists a constant $R \in (0,1)$ such that

$$\frac{\tau R}{f(R)} - \frac{\tau}{\ln(1-\tau')} \int_0^R \frac{d\sigma}{f(\sigma)} < \frac{\varepsilon}{2}. \quad (36)$$

Since u quenches at the time T_q, there exist

$T_1 \in (T_q - \frac{\varepsilon}{2}, T_q)$ and $h_0(\varepsilon) > 0$ such that $0 < u_{\min}(t) < \frac{R}{2}$ for $t \in [T_1, T_q), h \leq h_0(\varepsilon)$. Let q be a positive integer such that $t_q = \sum_{n=0}^{q-1} \Delta t_n \in [T_1, T_q)$

for $h \leq h_0(\varepsilon)$. It follows from Theorem 5.1 that

the problem (25) --(26) has a solution $U_h^{(n)}$ which obeys

$$\| U_h^{(n)} - u_h(t_n) \|_{\infty} < \frac{R}{2} \text{ for } n \leq q, h \leq h_0(\varepsilon)$$

which implies that

$$\| U_h^{(q)} \|_{\inf} \leq \| U_h^{(q)} - u_h(t_q) \|_{\infty} + \| u_h(t_q) \|_{\inf} < \frac{R}{2} + \frac{R}{2} = R, \quad h \leq h_0(\varepsilon).$$

From Theorem 5.2, $U_h^{(n)}$ quenches at the time $T_h^{\Delta t}$. It follows from Remark 5.1 and (36) that

$$|T_h^{\Delta t} - t_q| \leq \frac{\tau \| U_h^{(q)} \|_{\inf}}{f(\| U_h^{(q)} \|_{\inf})} - \frac{\tau}{\ln(1-\tau')} \int_0^{\| U_h^{(q)} \|_{\inf}} \frac{d\sigma}{f(\sigma)} < \frac{\varepsilon}{2}$$

because $\| U_h^{(q)} \|_{\inf} < R$ for $h \leq h_0(\varepsilon)$. We deduce that for

$$h \leq h_0(\varepsilon), | T_q - T_h^{\Delta t} | \leq | T_q - t_q | + | t_q - T_h^{\Delta t} | \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon,$$

which leads us to the result. □

6- Numerical result

In this section, we give some computational experiments to the quenching time for the solution of the problem (1)-(3) to confirm the theory developed in the previous section, Firstly, we take the explicit scheme in (37)-(38).

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \delta^2 U_i^{(n)} - \beta_i (U_i^{(n)})^{-p}, 0 \leq i \leq I, \quad (37)$$

$$U_i^{(0)} = \varphi_i > 0, \quad 0 \leq i \leq I, \quad (38)$$

where $n \geq 0, \Delta t_n = K \| U_h^{(n)} \|_{\inf}^{p+1}$ with $K = h^2$ and $p = 1$. In the case where,

$u_0(x) = \frac{0.99 + \varepsilon \sin(\pi x)}{2}$ with $0 < \varepsilon \leq 1$ and $\beta_i = 1 - 0.1\varepsilon \sin(i\pi h)$.

Secondly, we use the following implicit scheme in (39)-(40).

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \delta^2 U_i^{(n+1)} - \beta_i (U_i^{(n)})^{-p-1} U_i^{(n+1)}, \quad 0 \leq i \leq I, \quad (39)$$

$$U_i^{(0)} = \varphi_i > 0, \quad 0 \leq i \leq I, \quad (40)$$

In the case where, $n \geq 0, \Delta t_n = K \| U_h^{(n)} \|_{\inf}^{p+1}$ with $K = h^2, \varphi_i = \frac{2 + \varepsilon \sin(\pi i h)}{4}, 0 \leq i \leq I$ and $\beta_i = 1 - \varepsilon \sin(i\pi h)$. For the above implicit scheme, the existence and positivity of thediscrete solution $U_h^{(n)}$ is guaranteed using standard methods (see [6]). In the tables 1-10, in rows, we present the numerical quenching times, the numbers of iterations and the CPU times corresponding to meshes of 16, 32, 64, 128. We take for the numerical quenching time $t_n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when $\Delta t_n = | t_{n+1} - t_n | \leq 10^{-16}$.

Table 1:

Numerical quenching times, numbers of iterations and CPU times (seconds) obtained with the explicitEuler method for $\varepsilon = 1$

I	t_n	n	CPU time
16	0.312643	4161	1
32	0.312724	16023	3
64	0.312852	61257	60
128	0.312795	235525	1245

Table2:Numerical quenching times,numbers of iterations and CPU times (seconds) obtained with the implicit Euler method for $\varepsilon = 1$

I	t_n	n	CPU time
16	0.12683	3891	2
32	0.126653	14913	17
64	0.126611	56979	23
128	0.126600	217121	98

Table3: Numerical quenching times,numbers of iterations and CPU times (seconds) obtained with the implicit Euler method for $\varepsilon = \frac{1}{100}$.

I	t_n	n	CPU time
16	0.12683	3891	2
32	0.126653	14913	17
64	0.126611	56979	23
128	0.126600	217121	98

Table4: Numerical quenching times,numbers of iterations and CPU times (seconds) obtained with the implicit Euler method for $\varepsilon = \dfrac{1}{100}$

I	t_n	n	$CPU\ time$
16	0.127422	3901	2
32	0.126803	14928	11
64	0.126309	56979	33
128	0.126609	217134	109

Table5:Numerical quenching times,numbers of iterations and CPU times (seconds) obtained with the implicit Euler method for $\varepsilon = \dfrac{1}{1000}$

I	t_n	n	$CPU\ time$
16	0.125158	3887	2
32	0.125098	14864	16
64	0.124943	56779	222
128	0.124932	216314	3887

Table6:Numerical quenching times,numbers of iterations and CPU times (seconds) obtained with the implicit Euler method for $\varepsilon = \dfrac{1}{1000}$

I	t_n	n	$CPU\ time$
16	0.125902	3903	26
32	0.125286	14915	52
64	0.125131	56939	154
128	0.125093	216914	781

Table7:Numerical quenching times,numbers of iterations and CPU times (seconds) obtained with the implicit Euler method for $\varepsilon = \dfrac{1}{10000}$

I	t_n	n	$CPU\ time$
16	0.125158	3887	2
32	0.125098	14864	16
64	0.124943	56779	222
128	0.124932	216314	3887

Table8:Numerical quenching times,numbers of iterations and CPU times (seconds) obtained with the implicit Euler method for $\varepsilon = \dfrac{1}{10000}$

I	t_n	n	$CPU\ time$
16	0.125751	3903	21
32	0.125134	14914	54
64	0.124980	56793	122
128	0.124941	216327	887

Table9:Numerical quenching times,numbers of iterations and CPU times (seconds) obtained with the implicit Euler method for $\varepsilon = 0$

I	t_n	n	$CPU\ time$
16	0.125142	3887	2
32	0.125092	14864	11
64	0.124944	56778	133
128	0.125012	216314	98

Table10: Numerical quenching times,numbers of iterations and CPU times (seconds) obtained with the implicit

Euler method for $\varepsilon = \dfrac{1}{10000}$

I	t_n	n	$CPU\ tin$
16	0.125734	3903	3
32	0.125117	14914	14
64	0.124980	56792	118
128	0.125033	216324	274

Remark 6.1

When $\varepsilon=0$ and $p=1$, we know that the quenching time of the continuous solution of (1)--(3) is equal 0.125. We have also

seen in Remark 3.3 that the quenching time of the semidiscrete solution is equal 0.125. We observe from Tables 1--10 that when

ε decays to zero, then the numerical quenching time of the discrete solution goes to 0.125. When one examines tables

1,2,3 and 4 one sees that an important perturbation on the potential and the initial datum has a meaningful impact on the numerical quenching time.

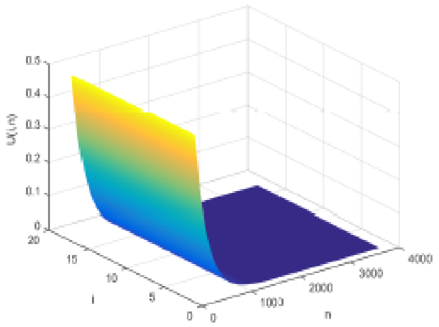

FIGURE 1 – Evolution of the discrete solution, source $f(U)=(U_i^{(n)})^{-p}$, $\varepsilon=1/10000$, $\varphi_i=\frac{0.99+\varepsilon\sin(\pi x)}{2}$, $I=16$ (implicit scheme).

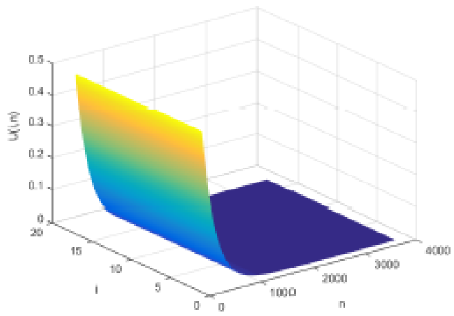

FIGURE 2 – Evolution of the discrete solution, source $f(U)=(U_i^{(n)})^{-p}$, $\varepsilon=1/10000$, $\varphi_i=\frac{0.99+\varepsilon\sin(\pi x)}{2}$, $I=16$ (explicit scheme).

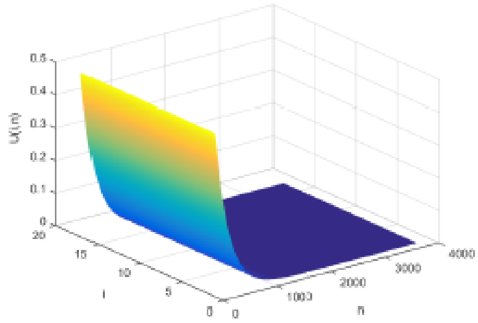

FIGURE 3 – Evolution of the discrete solution, source $f(U)=(U_i^{(n)})^{-p}$, $\varepsilon=1/10000$, $\varphi_i=\frac{0.99+\varepsilon\sin(\pi x)}{2}$, $I=32$ (implicit scheme).

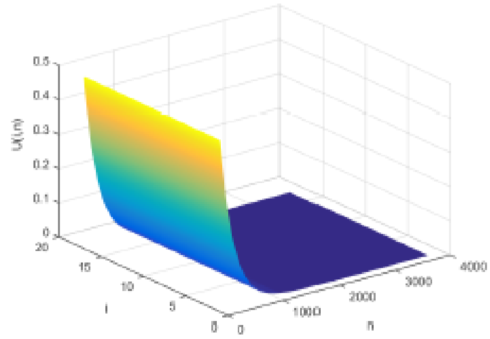

FIGURE 4 – Evolution of the discrete solution, source $f(U)=(U_i^{(n)})^{-p}$, $\varepsilon=1/10000$, $\varphi_i=\frac{0.99+\varepsilon\sin(\pi x)}{2}$, $I=32$ (explicit scheme).

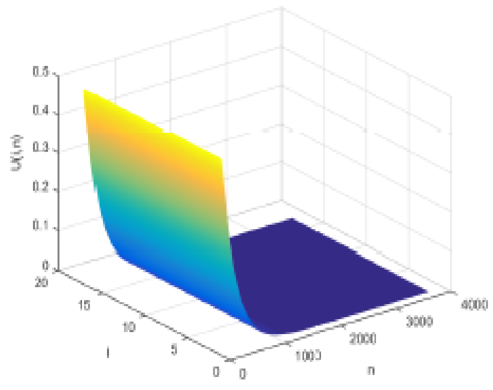

FIGURE 5 – Evolution of the discrete solution, source $f(U) = (U_i^{(n)})^{-p}$, $\varepsilon = 0$, $\varphi_i = \frac{0.99+\varepsilon\sin(\pi x)}{2}$, $I = 16$ (implicit scheme).

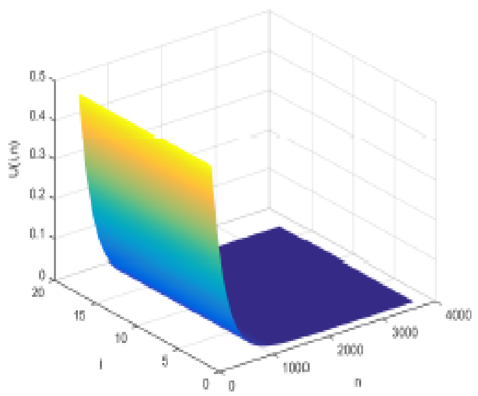

FIGURE 6 – Evolution of the discrete solution, source $f(U) = (U_i^{(n)})^{-p}$, $\varepsilon = 0$, $\varphi_i = \frac{0.99+\varepsilon\sin(\pi x)}{2}$, $I = 16$ (explicit scheme).

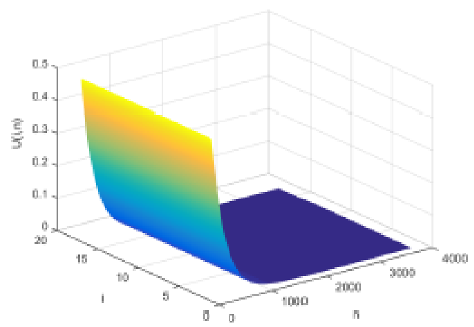

FIGURE 8 – Evolution of the discrete solution, source $f(U) = (U_i^{(n)})^{-p}$, $\varepsilon = 0$, $\varphi_i = \frac{0.99+\varepsilon\sin(\pi x)}{2}$,$I = 32$ (explicit scheme).

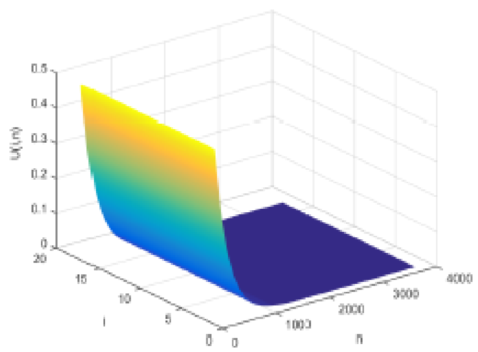

FIGURE 7 – Evolution of the discrete solution, source $f(U) = (U_i^{(n)})^{-p}$, $\varepsilon = 0$, $\varphi_i = \frac{0.99+\varepsilon\sin(\pi x)}{2}$, $I = 32$ (implicit scheme).

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