Products of Generalized Probabilistic Metric Spaces

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Abstract— In this paper we have studied and expanded the theory of generalized probabilistic metric spaces. Main emphasis is given on the results concerning the products of generalized probabilistic metric spaces.

Keywords—Generalized probabilistic metric space PG – equilateral space PG – Simple space.

I. Introduction

Probabilistic generalization of metric space was first introduced by Karl Menger [8] in 1942 and almost at the same time was studied by A. Wald [15]. The study of Probabilistic metric space expanded rapidly by pioneering works of Schweizer and Sklar [12] along with many other mathematicians (see the references). In 1963, S. Gähler [6], introduced the concept of 2-metric space which was supposed to be the generalization of usual notion of metric space. This concept was studied by many mathematicians and later generalized to probabilistic 2-metric space. Very recently, Mustafa and Sims [10] introduced a perfect deterministic generalization of metric space called generalized metric space.(G – Metric Space). On the pattern of Manger's probabilistic metric space, Zhou, Wang, Ciric and Alsulami [16] introduced probabilistic version of G – metric space. Keeping in view that it is the frontier branch between probability theory and functional analysis, we have tried to extend the theory of Probabilistic G – Metric Space in the present work.

II. Preliminaries

Definition 2.1: A distribution function is a real valued non-decreasing function F defined on R, with $F(-\infty) = 0$ and $F(-\infty) = 1$. The set of all distribution functions that are left continuous on $(-\infty, \infty)$ may be denoted as Δ.

For any $a \epsilon R$, the family of distribution functions ε_a may be defined as

$$\varepsilon_a(x) = \begin{cases} 0 \ , & x \leq a \\ 1 \ , & x > a \end{cases}$$

A distance distribution function (briefly, a d. d. f.) is a non-decreasing function F defined on R^+, that satisfies $F(0) = 0$ and $F(\infty) = 1$ and is left continuous on $(0, \infty)$.

For any distribution function F, and any $x > 0$,

$F\left(\frac{x}{0}\right) = 1$ and $F\left(\frac{0}{0}\right) = 0$.

Definition 2.2: A function $T: [0,1] \times [0,1]n$ to $[0,1]$ is called continuous t-norm, if it satisfies the following:

(T_1) $T(a, 1) = a$ for all $a \,\epsilon\, [0,1]$;
(T_2) T is commutative and associative, i.e. $T(a, b) = T(b, a)$ and $T(T(a, b), c) = T(a, T(b, c))$, for all a, b and $c \,\epsilon\, [0,1]$;
(T_3) T is continuous ;
(T_4) $T(a, b) \geq T(c, d)$ for $a \geq c$ and $b \geq d$ and for all a, b, c and $d \,\epsilon\, [0,1]$.

Here, we list some t-norms in order of decreasing "strength", where T'' is said to be stronger than T' if $T''(a, b) \geq T'(c, d)$ for all a, b, c and $d \,\epsilon\, [0,1]$ with strict inequality for at least one (a, b).

1. $M(a, b) = Min(a, b)$.
2. $P(a, b) = a \times b$.
3. $W(a, b) = Max(a + b - 1,0)$.

Recently, Mustafa and Sims [10] introduced a deterministic generalization of metric space called generalized metric space (G – metric space) as follows:

Definition 2.3: Let S be a non-empty set and $G: S \times S \times S \rightarrow R^+$ be a function satisfying the following conditions:

(G_1) $G(u, v, w) = 0$, if $u = v = w$ for all $u, v, w \,\epsilon\, S$;
(G_2) $0 < G(u, u, v)$ for all $u, v \,\epsilon\, S$ with $u \neq v$;
(G_3) $G(u, u, v) \leq G(u, v,$ for all $u, v, w \,\epsilon\, S$ with $w \neq v$;
(G_4) G is invariant under all permutations of u, v and w;
(G_5) $G(u, u, w) \leq G(u, a, a) + G(a, v, w)$ for all $u, v, w,$ and $a \,\epsilon\, S$.

Then G is called generalized metric or G – metric On S and the ordered pair (S, G) is called G – metric space.

On the pattern of Manger's probabilistic metric space, Zhou, Wang, Ciric and Alsulami [16] introduced probabilistic version of G – metric space.

Here, we discuss the probabilistic generalization generalized metric space in detail:

Consider the ordered pair $(S, \mathcal{T})$ consisting of non-empty set S and a mapping $\mathcal{T}$ from $S \times S \times S$ to Δ, collection of distribution functions. The value of $\mathcal{T}$ at (u, u, v) belonging to $S \times S \times S$ is represented by $G^*_{u,v,w}$. The function $G^*_{u,v,w}$ is assumed to satisfy the following conditions:

(P_1) $G^*_{u,v,w}(x) = 1$ for all $x > 0$ and for all $u, v, w \in S$ if and only if $u = v = w$;

(P_2) $G^*_{u,u,v}(x) \geq G^*_{u,v,w}(x)$ for all $x > 0$ and for all $u, v, w \in S$ with $v \neq w$;

(P_3) $G^*_{u,v,w}(x) = G^*_{u,w,v}(x) = G^*_{v,u,w}(x) = \cdots$ (Symmetry in all three variables)

(P_4) If $G^*_{u,a,a}(x) = 1$ and $G^*_{a,v,w}(y) = 1$ for all $y > 0$, u, v, w and $a \in S$. Then $G^*_{u,v,w}(x + y) = 1$.

In every $G -$ metric space (S, G) , metric G induces a mapping $\mathcal{T}: S \times S \times S \to \Delta$ such that $\mathcal{T}(u, v, w)(x) = G^*_{u,v,w}(x) = \varepsilon_0(x - G(u, v, w))$ for every triplet (u, u, v) in $S \times S \times S$.

Clearly:

(i) $G^*_{u,v,w}(x) = \varepsilon_0(x - G(u, v, w))$

$= \varepsilon_0(x)$ if and only if $u = v = w$

$\Rightarrow G^*_{u,v,w}(x) = 1$, for all $x > 0$ if and only if $u = v = w$.

(ii) $G^*_{u,u,v}(x) = \varepsilon_0(x - G(u, u, v))$

$\geq \varepsilon_0(x - G(u, v, w))$

[Since $G(u, u, v) \leq G(u, v, w)$ when $w \neq v$]

$\Rightarrow G^*_{u,u,v}(x) \geq G^*_{u,v,w}(x)$, for all $u, v, w \in S$ with $w \neq v$;

(iii) Proof of (P_3) is straight forward, so we omit the details.

(iv) $G^*_{u,a,a}(x) = 1 \Rightarrow x - G(u, a, a) > 0$

(v) $\Rightarrow G(u, a, a) < x$

$Ga, v, w * (y) = 1 \Rightarrow y - G(a, v, w) > 0$

$\Rightarrow G(a, v, w) < y$

$G(u, u, w) \leq G(u, a, a) + G(a, v, w) \Longrightarrow G(u, v, w) < x + y$

Thus, $G^*_{u,v,w}(x + y) = \varepsilon_0\big(x + y - G(u, v, w)\big) = 1$.

In fact (P_4) is a minimal generalization of triangle inequality, which may be interpreted as: "Perimeter of the triangle with vertices u, a, a, is less than x and perimeter of the triangle with vertices a, v, w is less than y then perimeter of the triangle with vertices u, v, w must be certainly less than x +y". However the most appropriate probabilistic generalization of triangle inequality of G – metric space was introduced in [17].

(P_5) $G^*_{u,v,w}(x + y) \geq T(G^*_{u,a,a}(x), G^*_{a,v,w}(y))$ for all $x, y > 0$ and $u, v, w, a \in S$.

We call the ordered pair $(S, \mathcal{T})$ as probabilistic $G -$ metric space (briefly PGM space) if (P_1) , (P_2) , (P_3) and (P_4) are satisfied. $(S, \mathcal{T}, T)$ is called Menger probabilistic $G -$ metric space (briefly MPGM space) if (P_1) , (P_2) , (P_3) and (P_5) are satisfied.

Concept of neighborhoods in MPGM space is defined as follows:

Definition 2.4: Let, (S, G^*, T) be a Menger probabilistic $G -$ metric space and u_0 be any point in S . For any $\varepsilon > 0$ and λ with $0 < \lambda < 1$, an (ε, λ) neighborhood of u_0 is the set of all points v in S for which

$G^*_{u_0,v,v}(\varepsilon) \geq 1 - \lambda$ and $G^*_{v,u_0,u_0}(\varepsilon) \geq 1 - \lambda$.

In [16] topology, convergence and completeness have been discussed.

III. PRODUCT SPACES

Products of probabilistic metric spaces was first discussed by V. Istratescu and I Vaduva [7]. Later A. Xavier [15], S., Zertaj, and A., Khan [13] and many others studied the results concerning products of probabilistic metric as well as 2-metric spaces. In this section, we have tried to define and establish some results regarding products of probabilistic $G -$ metric spaces.

Definition 3.1: Let $(S_1, \mathcal{T}_1)$ and $(S_2, \mathcal{T}_2)$ be probabilistic $G -$ metric spaces (briefly PGM spaces) and let T be left – continuous t – norm. Then $T -$ product $(S_1, \mathcal{T}_1) \times (S_2, \mathcal{T}_2)$ of $(S_1, \mathcal{T}_1)$ and $(S_2, \mathcal{T}_2)$ Is the space $(S_1 \times S_2, T(\mathcal{T}_1, \mathcal{T}_2))$ where $S_1 \times S_2$ is the Cartesian product of S_1 and S_2 and $T(\mathcal{T}_1, \mathcal{T}_2)$ is the mapping from $(S_1 \times S_2) \times (S_1 \times S_2) \times (S_1 \times S_2)$ into the set of distribution functions Δ defined as

$$T(\mathcal{T}_1, \mathcal{T}_2)(u, v, w) = T(\mathcal{T}_1(u_1, v_1, w_1), \mathcal{T}_2(u_2, v_2, w_2))$$

for any $u = (u_1, u_2), v = (v_1, v_2), w = (w_1, w_2)$.

We shall denote $S_1 \times S_2$ by S and $T(\mathcal{T}_1, \mathcal{T}_2)$ by $\mathcal{T}_T$ and finally we omit the reference to T and write

$$\mathcal{T}_T(u, v, w) = G^*_{u,v,w}.$$

Following theorem is the immediate consequence of the Definition 3.1.

Theorem 3.1: $T -$ product $(S, \mathcal{T}_T)$ of two PGM space $(S_1, \mathcal{T}_1)$ and $(S_2, \mathcal{T}_2)$ is a PGM space.

Proof:

(P_1^*) $G^*_{u,v,w}(x) = \mathcal{T}_T(u, v, w)(x)$

$= T(\mathcal{T}_1(u_1, v_1, w_1), \mathcal{T}_2(u_2, v_2, w_2))$

$= T(G^*_{1u_1,v_1,w_1}(x), G^*_{2u_2,v_2,w_2}(x) = 1$

for all $x > 0$ and for all $u, v, w \in S$

If and only if $u_1 = v_1 = w_1$ and $u_2 = u_2 = w_2$

If and only if $u = v = w$;

(P_2^*) $G^*_{u,u,v}(x) = T\left(G^*_{1u_1,u_1,v_1}(x), G^*_{2u_2,u_2,v_2}(x)\right)$

$\geq T(G^*_{1u_1,v_1,w_1}(x), G^*_{2u_2,v_2,w_2}(x))$, with $v_1 \neq w_1$ and $v_2 \neq w_2$

$= G^*_{v,v,w}(x)$, for all $x > 0$ and for all $u, v, w \in S$ with $v \neq w$;

(P_3^*) $G^*_{u,v,w}(x) = G^*_{u,w,v}(x) = G^*_{v,u,w}(x) = \ldots\ldots\ldots$ (Symmetry in all three variables)

We omit details of proof as it is easy and straight forward,

(P_4^*) $G^*_{u,a,a}(x) = 1$ and $G^*_{a,v,w}(y) = 1$ for all $x, y > 0$ and u, v, w and $a \in S$. $\Rightarrow$

$T(G^*_{1u_1,a_1,a_1}(x), (G^*_{2u_2,a_2,a_2}(x)) = 1$

and $T(G^*_{1a_1,v_1,w_1}(y), (G^*_{2a_2,v_2,w_2}(y)) = 1$

$\Rightarrow G^*_{1u_1,a_1,a_1}(x) = G^*_{2u_2,a_2,a_2}(x) = (G^*_{1a_1,v_1,w_1}(y) = (G^*_{2a_2,v_2,w_2}(y) = 1$

$\Rightarrow G^*_{1u_1,v_1,w_1}(x+y) = 1$ and $G^*_{2u_2,v_2,w_2}(x+y) = 1$

$\Rightarrow G^*_{u,v,w}(x+y) = T(G^*_{1u_1,u_1,v_1}(x+y), (G^*_{2u_2,u_2,v_2}(x+y) = 1$.

Theorem 3.2: If $(S_1, \mathcal{T}_1, T)$ and $(S_2, \mathcal{T}_2, T)$are Menger probabilistic G – metric space under same left – continuous t – norm T, then their T – product is a Menger probabilistic G – metric space.

Proof: Here, we only need to prove

$$(P_5^*)\quad G^*_{u,v,w}(x+y) \geq T(G^*_{u,a,a}(x), G^*_{a,v,w}(y))$$
$$for\ all\ u,v,w,a \in S\ and\ x,y > 0.$$

$$G^*_{u,v,w}(x+y) = T\left(G^*_{1u_1,v_1,w_1}(x+y), G^*_{2u_2,v_2,w_2}(x+y)\right) \geq$$

$$T(T(G^*_{1u_1,a_1,a_1}(x), G^*_{1a_1,v_1,w_1}(x)), T(G^*_{2u_2,a_2,a_2}(y), G^*_{2a_2,v_2,w_2}(y))$$

(Using(P_5) and properties of t – norm)

$$\geq$$

$$T(T(G^*_{1u_1,a_1,a_1}(x), G^*_{2u_2,a_2,a_2}(y)), T(G^*_{1a_1,v_1,w_1}(x), (G^*_{2a_2,v_2,w_2}(y))$$
$$= T(G^*_{u,a,a}(x), (G^*_{a,v,w}(y)).$$

Corollary 3.1: If $(S_1, \mathcal{T}_1, T_1)$ and $(S_2, \mathcal{T}_2, T_2)$ are Menger probabilistic G – metric spaces, then their T – product is a Menger probabilistic G – metric space under t – norm T if T is left continuous norm weaker than T_1 and T_2.

We start our discussion regarding some particular spaces. The simplest G – metric spaces are equilateral spaces in which

$$f(x) = \begin{cases} G(u,v,w) = 0, & if\quad u = v = w \\ a > 0\quad, & otherwise \end{cases}$$

Accordingly, we define Probabilistic Generalized equilateral space (PG – equilateral space).

Definition 3.2: A PGM space $(S, \mathcal{T})$ is said to be PG – equilateral space if and only if there exists a G – metric on S and a distribution function D such that $D(0) = 0$ and $D(x) > 0$ for some $x > 0$ in which $\mathcal{T}$ is defined on $S \times S \times S$ by

$$\mathcal{T}(u,v,w)(x) = G^*_{u,v,w}(x) = \varepsilon_0(x), if\ u = v = w$$
$$\mathcal{T}(u,v,w)(x) = G^*_{u,v,w}(x) = D(x)\quad, otherwise.$$

Following result insures that product of PG – equilateral spaces is of same type under certain conditions.

Theorem 3.3: If $(S_1, \mathcal{T}_1)$ and $(S_2, \mathcal{T}_2)$ are PG – equilateral space generated by the same distribution function D, then their Min – product $(S_1 \times S_2, \mathcal{T}_{Min})$ is PG – equilateral space generated by the same distribution function D.

Proof: $G^*_{u,v,w}(x) = \mathcal{T}_T(u,v,w)(x)$

$$= Min(\mathcal{T}_1(u_1,v_1,w_1), \mathcal{T}_2(u_2,v_2,w_2))$$
$$= Min(G^*_{1u_1,v_1,w_1}(x), (G^*_{2u_2,v_2,w_2}(x))$$
$$for\ all\ x > 0$$
$$= Min(\varepsilon_0(x), \varepsilon_0(x))$$
$$= \varepsilon_0(x),\quad if\ u = v = w \in S$$

In all the cases of $u = (u_1, u_2)$, $v = (v_1, v_2)$, $w = (w_1, w_2)$ not all equal, the result follows with the remark that T – Min is a must.

$$G^*_{u,v,w}(x) = Min(G^*_{1u_1,v_1,w_1}(x), G^*_{2\,u_2,v_2,w_2}(x))$$
$$= Min(D(x), D(x)) = D(x)$$

(If u_1, v_1, w_1 are not all equal as well as u_2, v_2, w_2 are also not all equal)

$$G^*_{u,v,w}(x) = Min(\varepsilon_0(x), D(x))$$

(If $u_1 = v_1 = w_1$ but u_2, v_2, w_2 are not equal)

$$= Min(D(x), D(x)) = D(x)$$

Also, $G^*_{u,v,w}(x) = Min(D(x), \varepsilon_0(x))$

(If u_1, v_1, w_1 are not all equal but $u_2 = v_2 = w_2$)

$$= Min\ (D(x), D(x)) = D(x)$$

It's also necessary that both PG – equilateral spaces must be generated by the same distribution function D.

Now, we introduce the concept of Probabilistic Generalized simple spaces (PG – simple spaces), another class of particular spaces.

Definition 3.3: A PGM space $(S, \mathcal{T})$ is said to be PG – simple space if and only if there exists a G – metric on S and a distribution function D such that $D(0) = 0$ and $D(x) > 0$ for some $x > 0$ in which $\mathcal{T}$ is defined on $S \times S \times S$ by

$$\mathcal{T}(u,v,w)(x) = G^*_{u,v,w}(x) = \varepsilon_0(x)\quad if\ u = v = w,$$
$$\mathcal{T}(u,v,w)(x) = G^*_{u,v,w}(x) = D\left(\frac{x}{G(u,v,w)}\right), otherwise.$$

Theorem 3.4: PG – simple space is Menger probabilistic G – metric space for T = Min.

Proof: $(P_1)\quad G^*_{u,v,w}(x) = \varepsilon_0(x)\quad if\ u = v = w$

$$\Rightarrow G^*_{u,v,w}(x) = 1\ if\ u = v = w\ for\ all\ x > 0.$$

$$(P_2)\quad G^*_{u,u,v}(x) = D\left(\frac{x}{G(u,u,v)}\right)$$

if $and v\ are\ not\ equal$.

$$\geq D\left(\frac{x}{G(u,u,w)}\right)$$

$Since, G\ (u,u,v) \leq G(u,v,w) for\ all\ v \neq w \in S$

$$= G^*_{u,v,w}(x) for\ all\ x > 0.$$

$$(P_3)\quad G^*_{u,v,w}(x) = G^*_{u,w,v}(x) =$$
$$G^*_{v,u,w}(x) = \ldots\ldots\ldots (Symmetry\ in\ all\ variables)$$

$Its\ proof\ is\ straight\ forward\ so, we\ omit\ the\ details.$

$$(P_4)\quad For\quad G^*_{u,v,w}(x+y)$$
$$\geq T\ (G^*_{u,a,a}(x), G^*_{a,v,w}(y))$$

$we\ establish\ that$

$$D\left(\frac{(x+y)}{G(u,v,w)}\right) \geq min\left(D\left(\frac{x}{G(u,a,a)}\right), D\left(\frac{y}{G(a,v,w)}\right)\right)$$

$Traingle\ inequality\ of\ G - metric;$

$$G(u,v,w) \leq G(u,a,a) + G(a,v,w)$$

$$\Rightarrow \frac{x+y}{G(u,v,w)} \geq \frac{x+y}{G(u,a,a)+G(a,v,w)}\quad (1)$$

$Since\ G(u,a,a)\ and G(a,v,w)\ are\ positive,$

$$\text{Max}\left(\left(\frac{x}{G(u,a,a)}\right), \left(\frac{y}{G(a,v,w)}\right)\right) \geq \frac{x+y}{G(u,a,a)+G(a,v,w)} \geq$$

$$\text{Min}\left(\left(\frac{x}{G(u,a,a)}\right), \left(\frac{y}{G(a,v,w)}\right)\right)\quad (2)$$

Combining (1) and right hand inequality in (2)

$$\frac{x+y}{G(u,v,w)} \geq Min\left(\left(\frac{x}{G(u,a,a)}\right), \left(\frac{y}{G(a,v,w)}\right)\right)$$

$$\Rightarrow D\left(\frac{(x+y)}{G(u,v,w)}\right) \geq Min\left(D\left(\frac{x}{G(u,a,a)}\right), D\left(\frac{y}{G(a,v,w)}\right)\right)$$

$Since\ D\ is\ non - decreasing.$

Following lemma determines the condition under which product of G - metric is G - metric space.

Lemma 3.1: If (S_1, G_1) and (S_2, G_2) are G – metric spaces, then product $(S_1 \times S_2, Max\ (G_1, G_2))$ is G – metric space.

Proof: $Let\ us\ denote\ S_1 \times S_2\ by\ S\ and\ Max\ (G_1, G_2)\ by\ G.$

$Then,\quad G(u,v,w) = Max(G_1(u_1,v_1,w_1), G_2(u_2,v_2,w_2)$ $for\ all\ u = (u_1,u_2), v = (v_1,v_2), w = (w_1,w_2) \in S.$

(G_1) $G(u,v,w) = 0$
$\Leftrightarrow Max(G_1(u_1,v_1,w_1), G_2(u_2,v_2,w_2)) = 0$
$\Leftrightarrow u_1 = v_1 = w_1\ and\ u_2 = v_2 = w_2$
$\Leftrightarrow u = v = w, for\ all\ u,v,w \in S.$

(G_2) $G_1(u_1,u_1,v_1) > 0\ and\ G_2(u_2,u_2,v_2) > 0$
$\Rightarrow \mathrm{Max}(G_1(u_1,u_1,v_1), G_2(u_2,u_2,v_2)) > 0$
$\Rightarrow \mathrm{G(u,u,v))} > 0$, for all u, v, ∈ S, with $u \neq v$.

(G_3) $Max(G_1(u_1,u_1,v_1), G_2(u_2,u_2,v_2) < Max(G_1(u_1,v_1,w_1), G_2(u_2,v_2,w_2)$ (When $v_1 \neq w_1$ and $v_2 \neq w_2$)
$\Rightarrow G(u,u,v) \le G(u,v,w) for\ all\ u,v,w \in S\ with\ w \neq v.$

(G_4) It's easy to show that G is invariant under all permutations of u, v, and w so we omit the details.

(G_5) $G_1(u_1,v_1,w_1) \le G_1(u_1,a_1,a_1) + G_1(a_1,v_1,w_1)$ and
$(G_2(u_2,v_2,w_2) \le (G_2(u_2,a_2,a_2) + (G_2(a_2,v_2,w_2)$
$\Rightarrow Max(G_1(u_1,v_1,w_1), (G_2(u_2,v_2,w_2)) \le Max(((G_1(u_1,a_1,a_1) + G_1(a_1,v_1,w_1)), (G_2(u_2,a_2,a_2) + (G_2(a_2,v_2,w_2)))$.
$= Max((G_1(u_1,a_1,a_1), (G_2(u_2,a_2,a_2)) + Max(G_1(a_1,v_1,w_1), (G_2(a_2,v_2,w_2)$
$\Rightarrow G(u,v,w) \le G(u,a,a) + G(a,v,w)$
$for\ all\ u,v,w,a \in S.$

Theorem 3.5: If $(S_1, \mathcal{T}_1)$ and $(S_2, \mathcal{T}_2)$ are PG –simple spaces generated by G –metric spaces (S_1, G_1) and (S_2, G_2) respectively, and the same distribution function D, then their Min – product $(S_1 \times S_2, \mathcal{T}_{Min})$ is a PG – simple space generated by G – metric space $(S_1 \times S_2, Max\ (G_1, G_2))$ and D.

Proof: Let $u = (u_1,u_2), v = (v_1,v_2), w = (w_1,w_2) \in S_1 \times S_2$.
Using (P_1^*) of Theorem 3.1 it follows that

$$G^*_{u,v,w}(x) = \mathcal{T}_{Min}(u,v,w)(x)\ \ for\ all\ u,v,w \in S$$
$$= \mathrm{Min}\ (G^*_{1u_1,v_1,w_1}(x), G^*_{2u_2,v_2,w_2}(x))$$
$$= \varepsilon_0(x), \text{If and only if } u_1 = v_1 = w_1$$

If and only if
$$u = (u_1,u_2) = v = (v_1,v_2) = w = (w_1,w_2).$$

If u,v,w are not equal, we have the following alternatives:
(i) u ≠ v = w (ii) u = v ≠ w (iii) u = w ≠ v and (iv) u ≠ v ≠ w
For each alternative we have to prove that

$$G^*_{u,v,w}(x) = \mathcal{T}_{Min}(u,v,w)(x) = D\left(\frac{x}{G(u,v,w)}\right) \text{ for all } x > 0.$$

Here, $G(u,v,w) = \mathrm{Max}(G_1(u_1,v_1,w_1), (G_2(u_2,v_2,w_2))$
First let us consider (i) $u \neq v = w$.
We have following three possibilities in this case:
(a) $u_1 \neq v_1 = w_1,\ u_2 = v_2 = w_2$ (b) $u_1 = v_1 = w_1,\ u_2 \neq v_2 = w_2$ (c) $u_1 \neq v_1 = w_1, u_2 \neq v_2 = w_2$.

Let us consider,
(a) $u_1 \neq v_1 = w_1, u_2 = v_2 = w_2$

$$G^*_{u,v,w}(x) = \mathrm{Min}\left(D\left(\frac{x}{G_1(u_1,v_1,w_1)}\right), D\left(\frac{x}{G_2(u_2,v_2,w_2)}\right)\right)$$
$$= \mathrm{Min}\left(D\left(\frac{x}{G_1(u_1,v_1,w_1)}\right), \varepsilon_0(x)\right)$$
$$= D\left(\frac{x}{G_1(u_1,v_1,w_1)}\right),$$
$$= D\left(\frac{x}{\mathrm{Max}(G_1(u_1,v_1,w_1), 0)}\right)$$
$$= D\left(\frac{x}{\mathrm{Max}(G_1(u_1,v_1,w_1), G_2(u_2,v_2,w_2))}\right)$$
$$= D\left(\frac{x}{G(u,v,w)}\right) \text{ for all } x > 0 .$$

(b) In case of $u_1 = v_1 = w_1,\ u_2 \neq v_2 = w_2,$ we proceed as in(a) .
(c) $When\ u_1 \neq v_1 = w_1,\ u_2 \neq v_2 = w_2$

$$G^*_{u,v,w}(x) = Min\left(D\left(\frac{x}{G_1(u_1,v_1,w_1)}\right), D\left(\frac{x}{G_2(u_2,v_2,w_2)}\right)\right)$$
$$= D\left(\frac{x}{Max(G_1(u_1,v_1,w_1), G_2(u_2,v_2,w_2))}\right)$$
$$= D\left(\frac{x}{G(u,v,w)}\right) for\ all\ x > 0.$$

For the proof of other three cases,we proceed as in (i).

We will conclude the section with the discussion of topologies on product of Menger probabilistic G – metric spaces.

Theorem 3.6: Let $(S_1, \mathcal{T}_1, T)$ and $(S_2, \mathcal{T}_2, T)$ are Menger probabilistic G – metric space under same left – continuous t – norm T. Let $\mathcal{B}$' denotes the $\varepsilon - \lambda$ neighborhood system in $(S_1 \times S_2, \mathcal{T}_T, T)$ and $\mathcal{B}$ be neighborhood system in $(S_1 \times S_2, \mathcal{T}_T, T)$ consisting of cartesian product $N_{u_1} \times N_{u_2}$, where $N_{u_1} and\ N_{u_2}$ are $\varepsilon - \lambda$ neighborhoods in component spaces $(S_1, \mathcal{T}_1, T)$ and $(S_2, \mathcal{T}_2, T)$ respectively. Then neighborhood system $\mathcal{B}$' and $\mathcal{B}$ induces equivalent topologies in $(S_1 \times S_2, \mathcal{T}_T, T)$.

Proof: It has been established in[16] that system of $\varepsilon - \lambda$ neighborhoods $\mathcal{B}$ and $\mathcal{B}$' are bases for their respective topologies.
To prove that $\mathcal{B}$ and $\mathcal{B}$ induces equivalent topologies in $(S_1 \times S_2, \mathcal{T}_T, \mathrm{T})$, we show that for each B in $\mathcal{B}$ there exists B' in $\mathcal{B}$' such that B'⊆ B and conversely.
Let $A_1 \times A_2$ is in $\mathcal{B}$. Then there exists $\varepsilon_1 - \lambda_1$ neighborhood of u_1 i.e. $N_{u_1}(\varepsilon_1, \lambda_1)$ contained in A_1 and $\varepsilon_2 - \lambda_2$ neighborhood of u_2 i.e. $N_{u_2}(\varepsilon_2, \lambda_2)$ contained in A_2.

Let, $\varepsilon = Min(\varepsilon_1, \varepsilon_2)$, $\lambda = Min(\lambda_1, \lambda_2)$ and $u = (u_1, u_2)$. We will show that $N_u(\varepsilon, \lambda) \subseteq A_1 \times A_2$. Let $v = (v_1, v_2)$ belong to $N_u(\varepsilon, \lambda)$.Then , we have

$$G^* u_1, v_1, v_1(\varepsilon_1) = T(G^* u_1, v_1, v_1(\varepsilon_1), 1)$$
$$\geq T(G^* u_1, v_1, v_1(\varepsilon_1), G^* u_2, v_2, v_2(\varepsilon_2$$
$$\geq T(G^* u_1, v_1, v_1(\varepsilon), G^* u_2, v_2, v_2(\varepsilon))$$
$$= G^* u, v, v(\varepsilon)$$
$$> 1 - \lambda \geq 1 - \lambda_1.$$

Also, $G^* v_1, u_1, u_1(\varepsilon_1) = T(G^* v_1, u_1, u_1(\varepsilon_1), 1)$
$$\geq T\big(G^* v_1, u_1, u_1(\varepsilon_1), G^* v_2, u_2, u_2(\varepsilon_2)\big)$$
$$\geq T\big(G^* v_1, u_1, u_1(\varepsilon), G^* v_2, u_2, u_2(\varepsilon)\big)$$
$$= G^*_{v,u,u}(\varepsilon) > 1 - \lambda \geq 1 - \lambda_1.$$

This implies that $v_1 \in N_{u_1}(\varepsilon_1, \lambda_1)$.
Similarly, we can prove that $G^* u_2, v_2, v_2(\varepsilon_2) \geq 1 - \lambda_2$ and $G^* v_2, u_2, u_2(\varepsilon_2) \geq 1 - \lambda_2$.
Hence, $v_2 \in N_{u_2}(\varepsilon_2, \lambda_2)$. Thus, $N_u(\varepsilon, \lambda) \subseteq A_2 \times A_2$ and consequently B'⊆ B.
For converse, let $N_u(\varepsilon, \lambda) \in \mathcal{B}'$. Left continuity of T insures that $Sup_{x<1} T(x, x) = 1$.
This implies the existence of an η such that
$$T(1 - \eta, 1 - \eta) > 1 - \lambda.$$

Let $v = (v_1, v_2)$ belong to $N_{u_1}(\varepsilon, \eta) \times N_{u_2}(\varepsilon, \eta)$
Then,
$$G^*_{u,v,v}(\varepsilon) = T\left(G^*_{u_1, v_1, v_1}(\varepsilon), G^*_{u_2, v_2, v_2}(\varepsilon)\right),$$
$$\geq T(1 - \eta, 1 - \eta) > 1 - \lambda$$
and
$$G^*_{v,u,u}(\varepsilon) = T\left(G^*_{v_1, u_1, u_1}(\varepsilon), G^*_{v_2, u_2, u_2}(\varepsilon)\right)$$
$$\geq T(1 - \eta, 1 - \eta) > 1 - \lambda.$$
Thus, $v \in N_u(\varepsilon, \lambda)$ and hence
$$N_{u_1}(\varepsilon, \eta) \times N_{u_2}(\varepsilon, \eta) \subseteq N_u(\varepsilon, \lambda).$$
So, the proof is complete.

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