Modeling of the Gaussian laser beam by approximate solution of Helmholtz equation

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***Abstract*— The purpose of this research is the analysis of intensity distribution in the laser beam, both from experimental and theoretical point of view. I performed the measurement of intensity distribution across the beam of the He-Ne laser, and made an attempt to compare it with Gaussian distribution function. The analysis of the image profile can be used in determination the distance of the objects depending on the distribution of the spot contours of the laser beam with its width. As well as it can be used to capturing images to the objects and determining their distance and the shape of the objects. As we can see, the fit is very good. From the analysis of the fit it was found that (program which made the fit GNUPLOT-mention).**

Keywords—laser, Maxwell's equations, detector, Helmholtz equation

Introduction

Lasers are devices that amplify or increase the intensity of light to produce a highly directional, high-intensity beam that typically has a very pure frequency or wavelength. The beams come in sizes ranging from approximately one-tenth the diameter of a human hair to that of a very large building.[1] Since the generation of the first laser beam in 1960, the detection techniques have been developed in order to recognize and analyze properties of the beam. In general, the analysis of laser beam is based on energy measurement, the intensity distribution of the laser beam, beam divergence, waist parameter, number of modes and others [2]. In optics, a Gaussian beam is an example of electromagnetic wave whose transverse electric field and intensity distributions are well approximated by Gaussian functions. Many lasers emit beams that approximate a Gaussian profile, in which case the laser is said to be operating in the fundamental transverse mode. The Gaussian wave is commonly used in theoretical and experimental optics and its mathematical representation has successfully been applied *by* many workers, and the mathematical function that describes shape of the laser beam is approximate solution of Helmholtz equation. We get this approximation by solution of homogeneous wave equation, and wave equation can be derived from Maxwell's equations in empty space. Thus any solution of Maxwell's equations in empty space satisfies the wave equation.[3]

Solution of wave equation oscillating in time and Helmholtz equation

The wave nature of light can be seen in experiments on interference and diffraction. The light is theoretically depicted as electromagnetic wave fulfilling Maxwell equations. In contrast, the particle nature of light is expressed through the idea of a light quantum or photon resulting in the theoretical description from quantization of the electromagnetic field. However, the degree of coherence of laser light is much better than that of other forms of light, and it is only in exceptional cases that the quantization of the electromagnetic field of laser light manifests itself in any substantial effect. Therefore, we shall discuss the nature of light as expressed in terms of classical electromagnetic waves, and we will describe in detail the propagation of light, and in particular the highly directional (and mostly paraxial) light from a laser, starting from Maxwell's equations.[4]
As is known from electromagnetic theory, the electric field $\vec{E}$, magnetic field $\vec{H}$, magnetic flux density $\vec{B}$, electric flux density $\vec{D}$, electric current density $\vec{J}$, and charge density ρ, all of which may change as functions of coordinates (x, y, z) and time t, are related by Maxwell's equations:

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \ldots\ldots\ldots\ldots\ldots(1)$$

$$\nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}, \ldots\ldots\ldots\ldots(2)$$

$$\nabla.\vec{D} = \rho, \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$\nabla.\vec{B} = 0. \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Here ∇ is the vector operator with $\partial/\partial x$, $\partial/\partial y$ and $\partial/\partial Z$ as its x, y, and Z components, respectively. Let ε denote the electric permittivity, μ the magnetic permeability and σ the electric conductivity of medium. We have then
$\vec{D} = \varepsilon E$, $\vec{B} = \mu\vec{H}$, $\vec{J} = \sigma\vec{E}$. ….(5)
By using the polarization $\vec{P}$ and the permittivity in vacuo ε_0, we have
$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$……………… (6)
The electric susceptibility is given by:
$\vec{P} = \varepsilon_0 x\vec{E}$…………………… (7)

In general,$\vec{P}$ is proportional to $\vec{E}$ when the electric field is weak, but it is no longer proportional when $\vec{E}$ is strong. In addition, it does not always follow that$\vec{P}$ varies in time in accordance with the time variation of$\vec{E}$. The$\vec{P}$-field follows the$\vec{E}$ -field only if x is constant, i.e. independent on the frequency of external field. We shall assume later that the medium is dielectric so that σ = 0, and the permeability is $\mu = \mu 0 = 4\pi \times 10^{-7}$H/m.

Applying the curl operation to both sides of (1) and using (2) and$\vec{B} = \mu\vec{H}$, we obtain

∇x $(\nabla\times\vec{E}) = -\nabla\times\frac{\partial \vec{B}}{\partial t} = -\mu\frac{\partial}{\partial t}\nabla\times\vec{H} = -\mu\frac{\partial^2\vec{D}}{\partial^2 t}$. ……….(8)

According to vector calculus we have

$\nabla(\nabla \times \vec{E}) = \nabla(\nabla.\vec{E}) - \nabla^2\vec{E}$,…………………………….(9)

so that (8) can be written as

$\nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E} = -\mu\frac{\partial^2\vec{D}}{\partial^2 t}$ …………………………..(10)

Using (3.4) and$\vec{D} = \varepsilon\vec{E}$ we have

$\nabla.\vec{E} = \frac{1}{\varepsilon}\nabla\cdot\vec{D} = \frac{\rho}{\varepsilon}$ …………….(11)

But, since an electric charge produces only an electrostatic field and is irrelevant to electromagnetic waves in an optical medium, we can neglect it and putρ = 0. Therefore, we have $\nabla\cdot\vec{E}$ = 0 and equation (10) becomes:

$\nabla^2\vec{E} - \varepsilon\mu\frac{\partial\vec{E}}{\partial t} = 0$ …………………..(12)

This is the equation of waves propagating with the velocity v such that $v^2 = \frac{1}{\varepsilon\mu}$, and the velocity of light in vacuo is $c = \frac{1}{\sqrt{\varepsilon 0\mu 0}}$.

Using Fourier expansion one can express any waveform as superposition of harmonic waves . On the other hand . laser light is almost perfectly monochromatic. Therefore, we can express the time factor of monochromatic electromagnetic wave of frequency $\omega$by $\vec{E}=\vec{\varepsilon}\exp(iwt)$, and the wave equation (12) becomes:

$\nabla^2\vec{\varepsilon} + k^2\vec{\varepsilon} = 0$, ……………….(13)

where $k^2 = \omega^2\varepsilon\mu$, and $k = \frac{\omega}{v}$ is the wave number.

In the ordinary treatment of wave optics it is sufficient to use the wave equation for a scalar variable u,

$\nabla^2 u + k^2 u = 0$. ……………...(14)

This equation is known as Helmholtz equation. For the explanation of diffraction, interference, birefringence, etc. This is equivalent to talking only of one of the components of the vector. In general, when dimensions of the medium are large compared to the wavelength, the light wave is almost purely transverse so that a scalar treatment is quietest is factory. We shall use Helmholtz equation *to* analyze the characteristics of monochromatic beam of light, It is well known that any arbitrary electromagnetic field can be expanded into plane or spherical waves, but a light beam propagating along an arbitrary axis can be approximately expanded into modes of Hermit-Gaussian functions along that axis [5].

Taking the z axis along the light beam and the wave number of the medium for transverse waves as k, we put

$u = A(x, y, z)\exp(-ikz)$ ……………… (15)

The function A representing the light beam must become practically zero for large values of x or y and changes slowly with z. Substituting equation (3.15) in (3.14) we get:

$(\frac{\partial^2 A}{\partial x^2})\exp(-ikz) + (\frac{\partial^2 A}{\partial y^2})\exp(-ikz) + (\frac{\partial^2 A}{\partial z^2})\exp(-ikz)$
$-2ik(\frac{\partial A}{\partial z})\exp(-ikz) - k^2 A\exp(-ikz) + k^2 A\exp(-ikz) = 0$, ……..(16)

which reduces to

$(\frac{\partial^2 A}{\partial x^2}) + (\frac{\partial^2 A}{\partial y^2}) - 2ik(\frac{\partial A}{\partial z}) + (\frac{\partial^2 A}{\partial z^2}) = 0$ ………………………………… (17)

The term $\exp-(ikz)$ of Equation 15 accounts for the wave oscillation along the propagation direction. The dependence of A on 2 is of a different nature. It likely accounts for the slow decrease in the amplitude of the wave as the wave propagates. Thus we can say that A varies slowly with z, and thus we can neglect the term $(\frac{\partial^2 A}{\partial Z^2})$ in front of the other ones and drop it from Equation (17) . The resulting equation is

$(\frac{\partial^2 A}{\partial x^2}) + (\frac{\partial^2 A}{\partial y^2}) - 2ik\left(\frac{\partial A}{\partial z}\right) = 0$ ………..(18)

Equation (18) is called the paraxial wave equation.

Approximate solution of the Helmholtz equation

A simple solution to the wave equation is one where we insert the simplest possible form of the solution and find the exact form that obeys the wave equation. The more formal solution is one where we just solve the wave equation in its full generality. We guess the simple trial solution to (18) to be of the form

$A(\vec{r}) = F_1(z)\exp[-\frac{\rho^2}{F2(Z)}]$. …………………………… (19)

Here $F_1(z)$ and $F_2(z)$ are slowly varying functions of z only, and $\rho^2 = x^2 + y^2$. To find the equations which describe$F_1(z)$ and $F_2(z)$ we will substitute the equation (3.19) into the equation (3.18). The first derivative of function A with respect to x is given by

$\frac{\partial A}{\partial x} = \quad F_1(z) \quad \left[-\frac{2x}{F_2(Z)}\right] \exp[-\frac{\rho^2}{F_2(Z)}]$ ……(20)

and the second derivative

$\frac{\partial^2 A}{\partial x^2} = F_1(Z)\left[-\frac{2}{F_2(z)}\right] exp\left[-\frac{\rho^2}{F_2(z)}\right] +$
$F_1\ (z)(-\frac{2x}{F_2(z)})^2 \exp[-\frac{\rho^2}{F2(z)}]$,….(21)

and similarly for the derivative with respect to y

$\frac{\partial^2 A}{\partial y^2} = F_1(Z)\left[-\frac{2}{F_2(z)}\right] exp\left[-\frac{\rho^2}{F_2(z)}\right] +$
$F_1\ (z)(-\frac{2y}{F_2(z)})^2 \exp[-\frac{\rho^2}{F2(z)}]$, ….(22)

According to equation (18) we will also find the first derivative of with respect to z

$\frac{\partial A}{\partial z} = F'_1(z) exp\left[-\frac{\rho^2}{F_2(z)}\right] +$
$exp[-\frac{\rho^2}{F_2(z)}$……………………….. (23)$F_1(z)\frac{\rho^2}{F_2(z)^2}F_2'(z)$

Substituting equations (21), (22), (23) into equation 3.18) we get

$-\frac{4F_1(z)}{F_2(z)} + F_1(z)\frac{4}{F_2^2(z)}(x^2+y^2) - 2ikF_1'(z) -$
$2ik\frac{F_1(z)}{F_2^2(z)}F_2'(x^2+y^2) = 0$,..(24)

or

$[-\frac{4F_1(z)}{F_2(z)} - 2ikF_1'(z) + (x^2+y^2)[-\frac{4F_1(z)}{F_2^2(z)} -$
$2ik\frac{F_1(z)}{F_2^2(z)}F_2'(z)] = 0$ ……….(25)

Equation (25) will be fulfilled if

$\frac{2F_1(z)}{F_2(z)} + ikF_1'(z) = 0$ ………..(26)

and

$\frac{2F_1(z)}{F_2^2(z)} - ik\frac{F_1(z)}{F_2^2(z)}F_2'(z)] = 0$ … (27)

separately. It follows from (27) that

$F_2'(z) = \frac{2}{ik}$ ……………..… (28)

By integration of the equation (28) we get:

$F_2(z) = \frac{2z}{ik} + c$ ……………….……(29)

where c is constant of integration.

We can rewrite the equation (26) as:

$ikF_1'(z) = -\frac{2F_1(z)}{F_2(z)}$ ………(30)

or

$\frac{F_1'(z)}{F_1(z)} = -\frac{2}{ik}\frac{1}{F_2(z)}$. ….(31)

By using the equation (29) in the equation (31) we obtain

$\frac{F_1'(z)}{F_1(z)} = -\frac{2}{ik\frac{2k}{ik}+c}$ · ……………..(32)

The last equation can be writ ton as

$\frac{d}{dz}\ln F_1(z) = -\frac{1}{Z+\frac{ik}{2}C}$, ……………….(33)

then

$\ln F_1(z) = -\ln(z + \frac{ik}{2}c) + c_1$, … … … …(3.34)

where c_1 can be written as $c_1 = \ln B_1$.

Then we can write equation (34) in the form

$F_1(Z) = \frac{B_1}{z+\frac{ikc}{2}}$·………………………….(35)

Modeling of the Gaussian laser beam by Approximate solution of Helmholtz equation

For the purpose of modeling of the Gaussian laser beam via approximate solution of Helmholtz equation we substitute the forms of F_1 (z) and $F_2(z)$, equations (35) and (29) respectively, into equation (19) and we get

$A(\vec{r}) = \frac{B_1}{Z+\frac{ikc}{2}} \exp\left[-\frac{\rho^2}{\frac{z_2}{ik}+c}\right]$, …..(36)

This equation represents the propagation of Gaussian beam in z direction. At each value of z the intensity is a Gaussian function of the radial distance ρ. This is why the wave is called a Gaussian beam. The Gaussian function has its peak at ρ= 0 (on axis) and drops monotonically with increasing ρ. The beam radius w(z) of the Gaussian distribution increases with the axial distance z. A large beam divergence for a given beam radius corresponds to poor beam quality. A low beam divergence can be important for applications such as pointing or free-space optical communications.[6]

Measurement of the intensity distribution in He-Ne laser

The theoretical (TEM00) beam has a perfect Gaussian profile. Lasers can produce many other TEM modes. In general, one can say that laser beams have a symmetric intensity profile. i.e. if we run across the beam, the intensity is minimum at the edge and as we move towards the center it increases and is maximum at the center and then it falls in a similar fashion as on the other side. For the purpose of present measurement we used the beam of the He-Ne laser (λ= 562.8 nm) with power (5 mW) and a high speed silicon detector to measure the intensity distribution of the laser light falling onto the detector when the laser beam is not attenuated. The detector was located 0.4m away from the output end of the laser and it moved on a translational stage in increments of mm, as show in Figure (1).

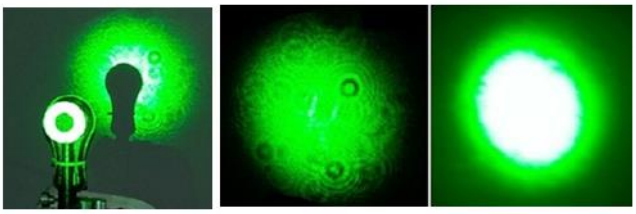

Figure 1: Scanning detector along diameter beam

The data were plotted and compared with the theoretically expected Gaussian form. The theoretical

Gaussian distribution has three parameters: I0 maximum intensity, x0-center of the beam (point of maximum intensity), and w-width or radius of the beam (1/2 the diameter). The fit of Gaussian curve to measurement data was with the use of GNUPLOT. For each measured point the square of the difference between theoretical fit and measured value was saved in an "error" cell, and the sum of these values was displayed to show the "quality of fit." I was able to adjust the theoretical graph constants until quality of the fit was nearly perfect (the lower value is the better fit).

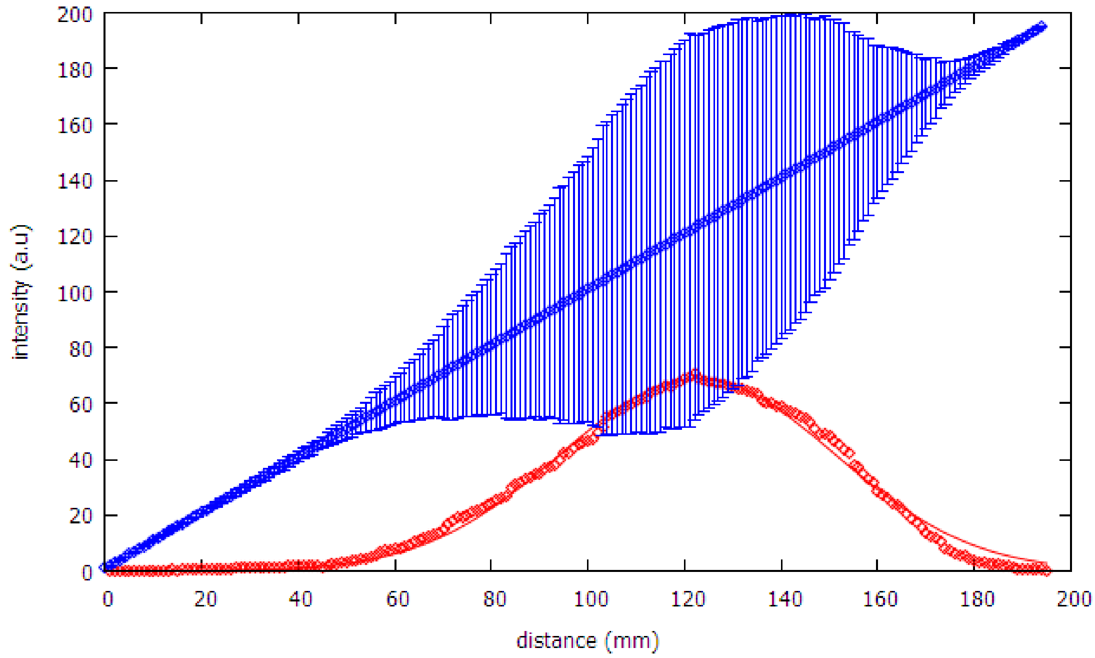

Figure 2: Results of measurement and fit to the Gaussian curve with distance x-axes.

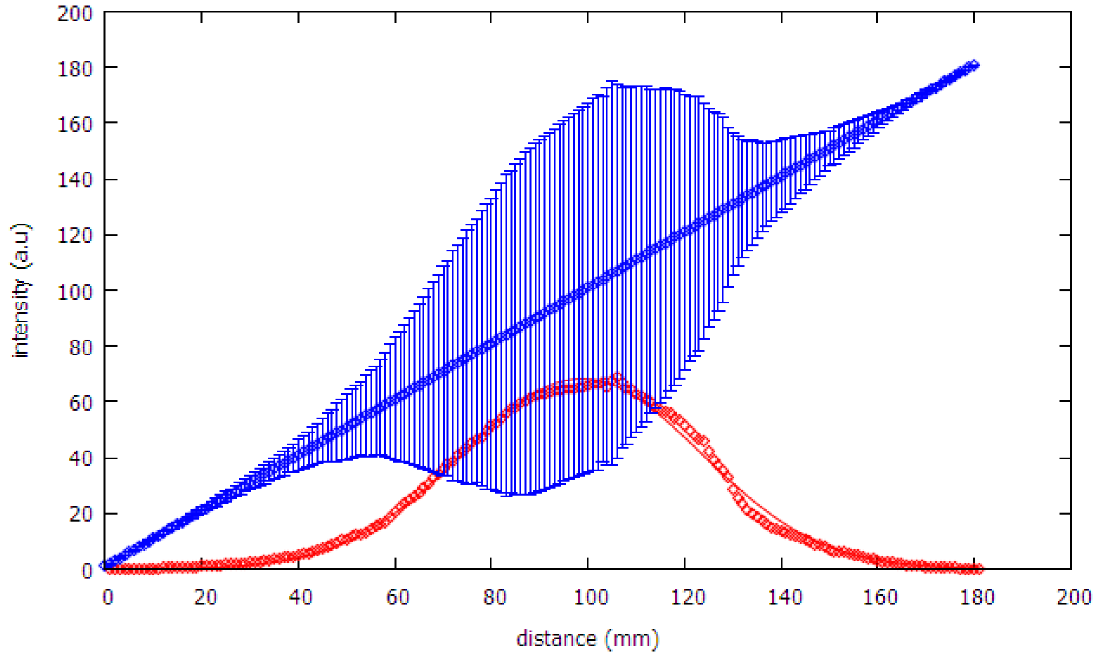

Figure 3: Results of measurement and fit to the Gaussian curve with distance y-axes.

Final set of parameters Asymptotic Standard Error
x0 = 104:484 +/- 0.3436 (0.3289%).
w = 70:3211 +/- 0.6883 (0.9788%).
I 0 = 147:568 +/- 1.249 (0.8463%).

Conclusions

- The mathematical equation which describes the Gaussian beam can be obtained from the approximate solution of the Helmholtz equation which follows from the wave equation.

- In this paper we have presented an exact solution for the Gaussian vectorial wave. Our solution satisfies Maxwell's equations. This exact solution has been compared to results obtained previously by other workers, notably the paraxial scalar and paraxial vectorial approximations.
- The topic of Gaussian beams provides students with the fundamentals for understanding the physics of laser beam and their propagation. Given the wide-spread use of lasers today, this material should be an essential part of course on optics. The coverage of high-order Gaussian modes serves to deepen the discussion of light waves and underscore the key components of the wave function: amplitude and phase.
- The beam quality with Gaussian shape (controlling the uniformity of the top hat and determination of the laser beam energy) is very important in medical applications especially in surgical operations. The intensity distribution of the laser beam is related to the source power and can be used in determination the distance of the objects.

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