Results For The Blowing-Up Phenomenon With A Localized Non-Linear Source Term

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Abstract. **This paper concerns the study of the numerical approximation for the following initial boundary value problem:**

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + f(u(1,t)), \quad (x,t) \in (-1,1)\times(0,T), \\ u(-1,t) = 0, \quad u_x(1,t) = 0, \quad t \in (0,T), \\ u(x,0) = u_0(x) \geq 0, \quad x \in [-1,1], \end{cases}$$

Where f(s) is a positive, increasing, convex function for the non-negative value of s,

$$\int_0^{+\infty} \frac{ds}{f(s)} < +\infty$$

$u_0(-1)=0$, $u_0'(1)=0$.

We find some conditions under which the solution of a discrete form of the above problem blows up in a finite time and estimate its numerical blow up time. We also prove the convergence of the numerical blow up time to the theoretical one. Finally, we give some numerical results to illustrate our analysis.

***Key words: discretization, blow-up time, localized, nonlinear reaction*.**

I. INTRODUCTION

Consider the following initial-boundary value problem

$$u_t(x,t) = u_{xx}(x,t) + f(u(1,t)), \quad (x,t) \in (-1,1)\times(0,T), \qquad (1)$$

$$u(-1,t) = 0, \quad u_x(1,t) = 0, \quad t \in (0,T), \qquad (2)$$

$$u(x,0) = u_0(x) \geq 0, x \in [-1,1], \qquad (3)$$

Which models the temperature distribution of a large number of physical phenomenon's from physics, chemistry and biology? The particularity of the problem described in (1)-(3) is that it represents a model in physical phenomenon where the reaction is driven by the temperature at a single site.

This kind of phenomenon is observed in biological and chemical diffusion processes in which the reaction takes place only at some local sites. This model is appropriate to describe:

(i) The influence of defect structures on a catalytic surface.

(ii) The temperature in a solid-fuel combustion scenario where the heat that is input into the system is localized, say as in a laser focused on one spot in the domain.

(iii) Chemical reaction-diffusion processes in which, due to effect of catalyst, the reaction takes place only at a single site.

(iv) A heat stationary source, which can support an explosive reaction. A stationary source provides a continuous supply of heat to the same environment.

(v) The ignition of a combustible medium with damping, where either a heated wire or a pair of small electrodes supplies a large amount of energy to every confined area.

For more motivation that is physical, *see [8]*

$$(A_0) \quad u_0 : [-1,1] \to [0,\infty)$$

is a positive, non-decreasing c^1 function.

$$u_0(-1) = 0, \quad u_0'(1) = 0,$$

$$(A_f) \quad f : [0,\infty) \to [0,\infty)$$

is a positive, increasing, convex function for the non negative value of s, C^1 function $\int_0^{+\infty} \frac{ds}{f(s)} < +\infty$.

Here (0,T) is the maximal interval of existence of the solution u.

The time T may be finite or infinite. . When T is infinite, we say that the solution u exists globally. When T is finite, the solution u develops a singularity in a finite time, namely :

$$\lim_{t\to T}\| u(.,t)\|_{\infty} = +\infty$$

Where

$$\| u(.,t)\|_{\infty} = \max_{-1\le x\le 1} |u(x,t)|$$

In this case, we say that the solution u blows up in a finite time and T is called the blow-up time of the solution u.
This kind of the phenomenon where the solution s of localized non linear heat equation with blow-up in a finite time have been the subject of investigation of many authors *(see [17],[44],[46])* and references cited therein.
In particular, the above problem has been studied and existence and uniqueness of classical solution has been proved. Under some assumptions, it is also shown that, the classical solution blows up in a finite time and its blow up time has been estimated (see *[17], [44], [45]).*

In this article, we are interesting in the numerical study of the above problem; finally, we show that under some assumption, the solution of a discrete form of $(1)-(3)$ blows up in a finite time and estimate its numerical blow up time. We also show that the numerical blow-up time converges to the real one when the mesh size goes to zero. At the end of the paper, we have shown how one may treat the case of Dirichlet boundary conditions. One may find in *[1],[7],[12],[13],[16],[18], [28],[31],[33]* ,[41] similar studies concerning other parabolic problems.

Let us notice that many authors have used numerical methods to study the phenomenon of blow-up but there are only a few studies on the convergence of the numerical blow-up time for solutions, which blow up in L^{∞} norm.
For instance in [2], the authors have proved the convergence of numerical blow-up time for solutions which blow up in L^{p} norm with $1 < p < \infty$.
In the next section, we give some results which will be used later. In the third section, under some assumptions, we show that the solution of a discrete form of the problem (1)-(3) blows up in a finite time and estimate its numerical blow-up time. In the fourth section, we show that, under some additional hypothesis, the numerical blow-up time goes to the real one when the mesh size goes to zero. Finally, we give some numerical results to illustrate our analysis

II. THE DISCRETE BLOW-UP SOLUTION

In this section, we study the phenomenon of blow-up using a full discrete explicit scheme of (1)-(3)

We start by the construction of a scheme as follows.
Let I be a positive integer, and $h = \frac{1}{I}$ define the grid $x_0=ih$ or $x_{i+1} = x_i + h$ and $\Delta t_n = x_{i+1} - x_i$ the step size,

φ_i is the discrete approximation of the initial data:

(A_{φ_i}) $\varphi_i : [0, I] \to [0,\infty)$
is a positive function and that

$u_i^{(n)}$ Approximates the solution $u(x_i,t_n)$ of the problem $(1)-(3)$

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + f(u(1,t)), (x,t) \in (-1,1)\times(0,T), & (1)\\ u(-1,t) = 0, \quad u_x(1,t) = 0, \quad t \in (0,T), & (2)\\ u(x,0) = u_0(x) \ge 0, \quad x \in [-1,1], & (3)\end{cases}$$

Approximate the problem (1)-(3) by the solution $U_h^{(n)} = (U_0^{(n)}, U_1^{(n)}, ..., U_I^{(n)})^T$,and approximate the initial condition $u_0(x)$ by $U_i^{(0)} = \varphi_i$ of the following discrete equations

$$\begin{cases} \delta_t U_i^{(n)} = \delta^2 U_i^{(n)} + f(U_i^{(n)}), \quad 1 \le i \le I & (4)\\ U_0^{(n)} = 0, & (5)\\ U_i^{(0)} = \varphi_i \ge 0, \quad 1 \le i \le I, & (6)\end{cases}$$

Where
$n \ge 0, \varphi_{i+1} \ge \varphi_i, 0 \le i \le I-1,$
With

$$\delta^2 U_i^{(n)} = \begin{cases} \dfrac{U_{i+1}^{(n)} - 2U_i^n + U_{i-1}^{(n)}}{h^2}, & 1 \le i \le I-1 \\ \dfrac{2}{h^2}(U_{I-1}^{(n)} - U_I^{(n)}), & i = I \end{cases}$$

$$\delta_t U_i^{(n)} = \frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n}$$ then

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} - \frac{U_{i+1}^{(n)} - 2U_i^n + U_{i-1}^{(n)}}{h^2} = f(U_i^{(n)}), \quad 1 \le i \le I-1$$
$$U_0^{(n)} = 0,$$
$$U_i^{(0)} = \varphi_i \ge 0, \quad 1 \le i \le I,$$
$$\frac{2}{h^2}(U_{I-1}^{(n)} - U_I^{(n)}) + \frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = f\left(U_I^{(n)}\right), \quad i = I$$

With

$$\Delta t_n = \min\left\{\frac{h^2}{3}, \frac{\tau}{f(\| U_h^{(n)}\|_\infty)}\right\}, \quad 0 < \tau < 1.$$

Let us notice that the restriction on the time step ensure the positivity of the discrete solution.
The following lemmas $1-5$ are preparatory results that will be used later.

Definition2.1
We say that the solution u of (1)-(3) blows up in a finite time, if there exist a finite time Tsuch that

$\| u(\cdot,t)\|_\infty < \infty$ for $t \in [0,T)$

but $\lim_{t\to T} \| u(\cdot,t)\|_\infty = \infty$ *where*

$\| u(\cdot,t)\|_\infty = \sup_{x\in\Omega} |u(x,t)|$.*and the time T is called the blow-up time of the solution u, when T is infinite, we say that the solution u exists globally.*

III PROPERTIEES OF THE DISCRETE SCHEME

In this section, we give some important results, which will be used later. The following lemmas 3.1-3.4 are a form the maximum principle for the discrete equations.

Lemma 3.1:

Let $U_h^{(n)}$ be the solution of $(4)-(6)$. *Then we have* $U_{i+1}^{(n)} \ge U_i^{(n)}, 0 \le i \le I-1.$

Proof:

Let $Z_i^{(n)} = U_{i+1}^{(n)} - U_i^{(n)}, \ 0 \le i \le I-1.$

We show that $Z_i^{(n)} \ge 0$ Obviously $Z_0^{(n)} \ge 0$ because $\varphi_{i+1} \ge \varphi_i$

A routine computation results that:

$$Z_i^{(n+1)} = U_{i+1}^{(n+1)} - U_i^{(n+1)}$$

$$\frac{Z_i^{(n+1)} - Z_i^{(n)}}{\Delta t_n} = \frac{Z_{i+1}^{(n)} - 2Z_i^{(n)} + Z_{i-1}^{(n)}}{h^2}, 1 \le i \le I-2,$$
$$\frac{Z_{I-1}^{(n+1)} - Z_{I-1}^{(n)}}{\Delta t_n} = \frac{-3Z_{I-1}^{(n)} + Z_{I-2}^{(n)}}{h^2}, \quad i = I-1$$

Which implies that:

$$Z_i^{(n+1)} = \frac{\Delta t_n}{h^2} Z_{i+1}^{(n)} + (1 - \frac{2\Delta t_n}{h^2}) Z_i^{(n)} + \frac{\Delta t_n}{h^2} Z_{i-1}^{(n)}, 1 \le i \le I-2,$$

and

$$Z_{I-1}^{(n+1)} = \frac{\Delta t_n}{h^2} Z_{I-2}^{(n)} + (1 - \frac{3\Delta t_n}{h^2}) Z_{I-1}^{(n)}.$$

Since $Z_i^{(0)} \ge 0, \ 1 \le i \le I-1,$

we deduce by induction $Z_i^{(n)} \ge 0, \ 0 \le i \le I-1$ and that the proof is complete.

The following lemma is a discrete form of the maximum principle.

Lemma 3.2:

Le $a^{(n)}$ be non-negative sequence and let $V_h^{(n)}$ be a sequence such that

$$\delta_t V_i^{(n)} - \delta^2 V_i^{(n)} - a^{(n)} V_i^{(n)} \ge 0, \ 1 \le i \le I, n \ge 0,$$
$$V_0^{(n)} \ge 0, \quad n \ge 0,$$
$$V_i^{(0)} \ge 0, \quad 0 \le i \le I.$$

Then $V_i^{(n)} \ge 0$ for $n \ge 0$; $0 \le i \le I$ if $\Delta t_n \le \frac{h^2}{3}$

Proof:
A routine calculation gives

$$\frac{V_i^{(n+1)} - V_i^{(n)}}{\Delta t_n} - \frac{V_{i+1}^{(n)} - V_i^{(n)} + V_{i-1}^{(n)}}{h^2} - a^{(n)} V_i^{(n)} \geq 0,$$

$1 \leq i \leq I-1,$

This implies that

$$\frac{V_i^{(n+1)} - V_i^{(n)}}{\Delta t_n} - \frac{V_{i+1}^{(n)} - 2V_i^{(n)} + V_{i-1}^{(n)}}{h^2} - a^{(n)} V_i^{(n)} \geq 0, 1 \leq i \leq I-1,.$$

Which implies that:

$$V_i^{(n+1)} \geq \frac{\Delta t_n}{h^2} V_{i+1}^{(n)} + (1 - 2\frac{\Delta t_n}{h^2}) V_i^{(n)} + \frac{\Delta t_n}{h^2} V_{i-1}^{(n)} + \Delta t_n a^{(n)} V_I^{(n)}, 1 \leq i \leq I-1,$$

$$V_I^{(n+1)} \geq \frac{2\Delta t_n}{h^2} V_{I-1}^{(n)} + (1 - 2\frac{\Delta t_n}{h^2}) V_I^{(n)} + \Delta t_n a^{(n)} V_I^{(n)}.$$

Since $\Delta t_n \leq \frac{h^2}{3}$, we deduce that $1 - 2\frac{\Delta t_n}{h^2}$ is nonnegative. Due the fact that $V_h^{(0)} \geq 0$, we deduce by induction that $V_h^{(n)} \geq 0$, for $n \geq 0$, which ends the proof.

A direct consequence of the above result is the following comparison lemma. Its proof is straightforward.

Lemma 3.3:

Suppose that $a^{(n)}$ and $b^{(n)}$ are two sequences such that $a^{(n)}$ is nonnegative.

Let $V_h^{(n)}$ and $W_h^{(n)}$ two sequences such that

$$\delta_t V_i^{(n)} - \delta^2 V_i^{(n)} - a^{(n)} V_i^{(n)} + b^{(n)} \leq \delta_t W_i^{(n)} - \delta^2 W_i^{(n)} - a^{(n)} W_i^{(n)} + b^{(n)},$$

$1 \leq i \leq I, n \geq 0,$

$V_0^{(n)} \leq W_0^{(n)},\ n \geq 0,$

$V_i^{(0)} \leq W_i^{(0)},\ 0 \leq i \leq I$

Then $V_i^{(n)} \leq W_i^{(n)}$ for $n \geq 0$

$0 \leq i \leq I$ if $\Delta t_n \leq \frac{h^2}{3}$

Now, let us give a property of operator δ_t

Let $f : [0, \infty) \to [0, \infty)$ be a C^1 convex function

Lemma 3.4:

Let $U_h^{(n)} \in$ □ be a sequence such that $U_h^{(n)} \geq 0$. Then we have

$\delta_t f(U_h^{(n)}) \geq f'(U_h^{(n)}) \delta_t U_h^{(n)}$ $n \geq 0$ and

$\delta^2 f(U_h^{(n)}) \geq f'(U_h^{(n)}) \delta^2 U_h^{(n)}$

Proof: We apply Taylor's expansion to obtain

$$f(U_h^{(n+1)}) = f(U_h^{(n)}) + (U_h^{(n+1)} - (U_h^{(n)})) f'(U_h^{(n)}) + \frac{(U_h^{(n+1)} - (U_h^{(n)}))^2}{2} f''(\theta)$$

Where θ is an intermediate value between $U_h^{(n+1)}$ and $U_h^{(n)}$

$$f(U_h^{(n-1)}) = f(U_h^{(n)}) + (U_h^{(n-1)} - (U_h^{(n)})) f'(U_h^{(n)}) + \frac{(U_h^{(n-1)} - (U_h^{(n)}))^2}{2} f''(\tau)$$

fWhere τ is an intermediate value between $U_h^{(n-1)}$ and $U_h^{(n)}$.

The first equation and the last imply that

$$f(U_h^{(n+1)}) - 2f(U_h^{(n)}) + f(U_h^{(n-1)}) = (U_h^{(n+1)} - 2U_h^{(n)} + U_h^{(n-1)}) f'(U_h^{(n)}) + \frac{(U_h^{(n+1)} - U_h^{(n)})^2}{2} f''(\theta) + \frac{(U_h^{(n-1)} - U_h^{(n)})^2}{2} f''(\tau)$$

$$\frac{f(U_h^{(n+1)}) - 2f(U_h^{(n)}) + f(U_h^{(n-1)})}{h^2} = \frac{(U_h^{(n+1)} - 2U_h^{(n)} + U_h^{(n-1)})}{h^2} f'(U_h^{(n)}) + \frac{(U_h^{(n+1)} - U_h^{(n)})^2}{2h^2} f''(\theta) + \frac{(U_h^{(n-1)} - U_h^{(n)})^2}{2h^2} f''(\tau).$$

The first equation and the lase one imply that

$$\delta^2 f(U_h^{(n)}) = f'(U_h^{(n)}) \delta^2 U_h^{(n)} + \frac{(U_h^{(n+1)} - U_h^{(n)})^2}{2h^2} f''(\theta) + \frac{(U_h^{(n-1)} - U_h^{(n)})^2}{2h^2} f''(\tau)..$$

And

$$\delta_t f(U_h^{(n)}) = f'(U_h^{(n)}) \delta_t U_h^{(n)} + \frac{1}{2} \Delta t_n \delta_t (U_h^{(n)})^2 f''(\theta)$$

Use the fact that $U_h^{(n)} \geq 0$ for $n \geq 0$ and using the convexity of f we obtain the desired result.

IV. BLOW-UP IN DISCRETE SOLUTION

In this section, under some assumptions, we show that the discrete solution blow up in a finite time and its numerical blow-up time converge to the real one when the mesh size tends to zero.

The following theorem shows that the discrete solution blows up under some conditions.

Definition 4.1

We say that the solution $U_h^{(n)}$ of (4)-(6) blows up in a finite time, if $\| U_h^{(n)}\|_\infty = +\infty$. *and*

$$T_h^{\Delta t} = \lim_{n\to\infty}\sum_{i=0}^{n-1}\Delta t_i < +\infty$$

The number $T_h^{\Delta t}$ is called the numerical blow-up time of the solution $U_h^{(n)}$

The following theorem shows that the discrete solution blows up under some conditions

***Theorem 4.1**:*
Suppose that there exists a positive constant $A\in(0,1]$ such that the initial that at (6) satisfies

$$\delta^2\varphi_i + f(\varphi_i)\ge 0, 0\le i\le I-1, \qquad (7)$$
$$\delta^2\varphi_I + f(\varphi_I)\ge Af(\varphi_I), \qquad (8)$$

With φ_i an approximation of the initial data.Then the solution $U_h^{(n)}$ of $(4)-(6)$ blows up in a finite time and its numerical blow-up time $T_h^{\Delta t}$ is estimated as follows.

$$T_h^{\Delta t}\le \frac{\tau}{f(\|\varphi_h\|_\infty)} + \frac{\tau}{\tau'}\int_{\|\varphi_h\|_\infty}^{+\infty}\frac{ds}{f(s)},$$

Where

$$\tau' = \min\{\frac{h^2}{3} f(\|\varphi_h\|_\infty),\tau\}.$$

Proof

Introduce the vector $J_h^{(n)}$ defined as follows

$$J_i^{(n)} = \delta_t U_i^{(n)},\quad 0\le i\le I-1,$$
$$J_I^{(n)} = \delta_t U_I^{(n)} - Af(U_I^{(n)})$$

By a straightforward computation we get

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} = \delta_t(\delta_t U_i^{(n)} - \delta^2 U_i^{(n)}),$$
$$0\le i\le I-1,\ n\ge 0,.$$

$$\delta_t J_I^{(n)} - \delta^2 J_I^{(n)} = \delta_t(\delta_t U_I^{(n)} - \delta^2 U_I^{(n)}) - A\delta_t f(U_I^{(n)})$$
$$+A\delta^2 f(U_I^{(n)})$$

Using(4), we obtain

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} = \delta_t f(U_i^{(n)}), 1\le i\le I-1,$$

$$\delta_t J_I^{(n)} - \delta^2 J_I^{(n)} = (1-A)\delta_t f(U_I^{(n)}) + A\delta^2 f(U_I^{(n)}).$$

It follows from lemma 2.4 that

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} \ge f'(U_i^{(n)})\delta_t U_i^{(n)}, 1\le i\le I-1,$$

$$\delta_t J_I^{(n)} - \delta^2 J_I^{(n)} \ge (1-A)f'(U_I^{(n)})\delta_t U_I^{(n)} + Af'(U_I^{(n)})\delta^2 U_I^{(n)}.$$

Taking into account(4), we deduce that

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} \ge f'(U_i^{(n)})\delta_t U_i^{(n)}$$
$$-Af'(U_i^{(n)})f(U_i^{(n)}) \qquad (9)$$
$$\delta_t J_I^{(n)} - \delta^2 J_I^{(n)} \ge f'(U_I^{(n)})\delta_t U_I^{(n)}$$
$$-Af'(U_I^{(n)})f(U_I^{(n)}). \qquad (10)$$

Which implies that

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} \ge f'(U_i^{(n)})J_i^{(n)}, 1\le i\le I. \ (11)$$

Obviously, we have $J_0^{(n)} = 0$.

From (7) we obtain $J_i^{(0)}\ge 0$

It follows from lemma 2.2 that $J_i^{(n)}\ge 0, 0\le i\le I$. Hence, we have

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n}\ge Af(U_I^{(n)}) \qquad (12)$$

Which implies that

$$U_I^{(n+1)}\ge U_I^{(n)} + A\Delta t_n f(U_I^{(n)}),$$

Since $U_I^{(n)} = \| U_h^{(n)}\|_\infty$ we arrive at

$$\| U_h^{(n+1)}\|_\infty \ge \| U_h^{(n)}\|_\infty + A\Delta t_n f(\| U_h^{(n)}\|_\infty) \qquad (13)$$

It is not difficult to see that

$$\Delta t_n f(\| U_h^{(n)}\|_\infty) = \min\{\frac{h^2}{3} f(|U_h^{(n)}\|_\infty),\tau\}.$$

From (14) we get

$\| U_h^{(n+1)}\|_\infty \ge \| U_h^{(n)}\|_\infty$ and by induction we obtain $\| U_h^{(n)}\|_\infty \ge \| U_h^{(0)}\|_\infty = \|\varphi_h\|_\infty$, which implies that

$$\Delta t_n f(\| U_h^{(n)}\|_\infty)\ge \min\{\frac{h^2}{3} f(\|\varphi_h\|_\infty),\tau\} = \tau'.$$

Consequently, we have

$$\|U_h^{(n+1)}\|_\infty \geq \|U_h^{(n)}\|_\infty + \tau' \quad (14)$$

Using a recursion argument, we discover that

$$\|U_h^{(n)}\|_\infty \geq \|U_h^{(0)}\|_\infty + n\tau' = \|\varphi_h\|_\infty + n\tau' \quad (15)$$

Since the term on the right hand, side of the above inequality tends to infinity as n approaches infinity. Hence, we see that $\|U_h^{(n)}\|_\infty$ goes to infinity as n approaches infinity.
Now let us estimate the numerical blow-up time.

From the restriction on the time step, we get:

$$\Sigma_{n=0}^{\infty}\Delta t_n \leq \Sigma_{n=0}^{\infty}\frac{\tau}{f(\|U_h^{(n)}\|_\infty)}.$$

Due to (14), we arrive at

$$\Sigma_{n=0}^{\infty}\Delta t_n \leq \Sigma_{n=0}^{\infty}\frac{\tau}{f(\|\varphi_h\|_\infty + n\tau')}.$$

We observe the

$$\int_0^{+\infty}\frac{ds}{f(\|\varphi_h\|_\infty + s\tau')} = \Sigma_{n=0}^{\infty}\int_n^{n+1}\frac{ds}{f(\|\varphi_h\|_\infty + s\tau')}$$

$$\geq \Sigma_{n=0}^{\infty}\frac{1}{f(\|\varphi_h\|_\infty + (n+1)\tau')}.$$

Since

$$\int_0^{+\infty}\frac{ds}{f(\|\varphi_h\|_\infty + s\tau')} = \frac{1}{\tau'}\int_{\|\varphi_h\|_\infty}^{+\infty}\frac{ds}{f(s)},$$

We deduce that

$$\Sigma_{n=0}^{\infty}\Delta t_n \leq \frac{\tau}{f(\|\varphi_h\|_\infty)} + \frac{\tau}{\tau'}\int_{\|\varphi_h\|_\infty}^{+\infty}\frac{ds}{f(s)}$$

Use the fact that the quantity on the right hand of the above inequality is finite to complete the rest of the proof

Remark 4.1:

From (13) $\|U_h^{(n+1)}\|_\infty \geq \|U_h^{(n)}\|_\infty + \tau'$. we *get by induction that*

$\|U_h^{(n)}\|_\infty \geq \|U_h^{(q)}\|_\infty + \tau'(n-q)$. *Hence*

$$T_h^{\Delta t} - T_q = \Sigma_{n=q}^{\infty}\Delta t_n \leq \Sigma_{n=q}^{\infty}\frac{\tau}{f(\|U_h^{(n)}\|_\infty)} \leq \Sigma_{n=q}^{\infty}\frac{\tau}{f(\|U_h^{(q)}\|_\infty + (n-q)\tau')}$$

where $t_q = \Sigma_{n=q}^{\infty}\Delta t_n$

We observe that

$$\int_0^{+\infty}\frac{ds}{f(\|U_h^{(q)}\|_\infty + s\tau')} = \Sigma_{n=0}^{\infty}\int_n^{n+1}\frac{ds}{f(\|U_h^{(q)}\|_\infty + s\tau')}$$

$$\geq \Sigma_{n=0}^{\infty}\frac{1}{f(\|U_h^{(q)}\|_\infty + (n+1)\tau')}.$$

Since

$$\int_0^{+\infty}\frac{ds}{f(\|U_h^{(q)}\|_\infty + s\tau')} = \frac{1}{\tau'}\int_{\|U_h^{(q)}\|_\infty}^{+\infty}\frac{ds}{f(s)},$$

We get

$$T_h^{\Delta t} - t_q \leq \frac{\tau}{f(\|U_h^{(q)}\|_\infty)} + \frac{\tau}{\tau'}\int_{\|U_h^{(q)}\|_\infty}^{+\infty}\frac{ds}{f(s)}.$$

Since $\tau' = \min\{\frac{h^2}{3}f(\|\varphi_h\|_\infty), \tau\}$, *if we take* $\tau = h^2$, *we get* $\frac{\tau}{\tau'} = \min\{\frac{1}{3}f(\|\varphi_h\|_\infty), 1\}$, *which implies that there exists positive constant* B *such that* $\frac{\tau}{\tau'} \leq B$.

V. CONVERGENCE OF THE BLOW-UP TIME

In this section, under some condition, we show that the discrete solution blows up in a finite time and that its numerical blow-up time goes to the analytic one when the mesh size goes to zero.
In order to prove the convergence of the discrete blow-up time, we need to show that the discrete scheme converges for each fixed interval time $[0,T]$.
We denote by $u_h(t_n) = (u(x_0,t_n),\ldots,u(x_I,t_n))^T$ and state the result on the convergence of our scheme by the following.

Theorem 5.1

Suppose that the problem (1)-(3) has a solution $u \in C^{4,2}([-1,1]\times[0,T])$, and that $U_h^{(n)}$ approximates the solution u of (1)-(3) with $U_h^{(0)} = \varphi_h$
Assume that the initial data at (6) verifies

$$\| \varphi_h - u_h(0)\|_\infty = o(1) \quad as\, h \to 0 \qquad (16)$$

Then the problem (5)-(8) *has a solution* $U_h^{(n)}$ *for* h *sufficiently small,* $0 \le n \le J$ *and we have the following estimate*

$$\max_{0\le n\le J} \| U_h^{(n)} - u_h(t_n)\|_\infty = O(\| \varphi_h - u_h(0)\|_\infty + h^2 + \Delta t_n) \text{ as } h \to 0$$

Where

J *is such that* $\Sigma_{n=0}^{J-1}\Delta t_n \le T$ *and* $t_n = \Sigma_{j=0}^{n-1}\Delta t_j$.

Proof:

For h, the problem (5)-(8) has a solution $U_h^{(n)}$. We want to prove that $U_h^{(n)}$ approaches to u_h as $h \to 0$. Let $N \le J$ be the greatest value of n such that

$$\| U_h^{(n)} - u_h(t_n)\|_\infty < 1 \quad \text{for} \quad n < N \qquad (17)$$

We know that $N \ge 1$ because of (15). Due to the fact $u \in C^{4,2}([-1,1]\times[0,T])$, there exists a positive constant K such that $\| u\| \le K$. Applying the triangle inequality, we obtain

$$\| U_h^{(n)}\|_\infty \le \| u_h(t_n)\|_\infty + \| U_h^{(n)} - u_h(t_n)\|_\infty \le 1 + K \qquad (18)$$

Since $u \in C^{4,2}([-1,1]\times[0,T])$, taking the derivative in x on both side of (1) and due to the fact that u_x, u_{xt}

vanish at x=1. we observe that u_{xxx} also vanish at x=1.

Using Taylor's expansion, we find that:

$$\delta_t u(x_i,t_n) - \delta^2 u(x_i,t_n) - f(u(x_I,t_n)) = -\frac{h^2}{12}u_{xxxx}(\tilde{x}_i,t)$$

$$\frac{\Delta t_n}{2}u_{tt}(x_i,\tilde{t}_n) \text{ for } 1 \le i \le I.$$

To establish the above equality for i=I, we have used the fact that at u_{xxx} vanishes at i=I, et $e_h^{(n)} = U_h^{(n)} - u_h(t_n)$ be the discretization error.

From the mean value theorem, we get

$$\delta_t e_i^{(n)} - \delta^2 e_i^{(n)} = f'(\varsigma_i^{(n)})e_i^{(n)} + \frac{h^2}{12}u_{xxxx}(\tilde{x}_i,t_n) - \frac{\Delta t_n}{2}u_{tt}(x_i,\tilde{t}_n), 1 \le i \le I$$

where $\varsigma_i^{(n)}$ is an intermediate value between $u(x_i,t_n)$ and $U_i^{(n)}$. Since $u_{xxxx}(x,t)$, $u_{tt}(x,t)$ are bounded, there exists a positive constant M such that,

$$\delta_t e_i^{(n)} - \delta^2 e_i^{(n)} \le f'(\varsigma_i^{(n)})e_i^{(n)} + M\Delta t_n + Mh^2, 0 \le i \le I \quad (19)$$

Let P=1+K and introduce the vector $V_h^{(n)}$ defined as follows

$$V_i^{(n)} = e^{(P+1)t_n}(\| \varphi_h - u_h(0)\|_\infty + Mh^2 + M\Delta t_n), 0 \le i \le I.$$

A straightforward computation gives:

$$\delta_t V_i^{(n)} - \delta^2 V_i^{(n)} \ge f'(\varsigma_i^{(n)})V_i^{(n)} + M\Delta t_n + Mh^2, 1 \le i \le I,$$

$$V_0^{(n)} \ge e_0^{(n)}, \quad V_i^{(0)} \ge e_i^{(0)}, \quad 0 \le i \le I$$

We observe that $f'(\varsigma_i^{(n)})$ is bounded above by f(P). It follows from comparison lemma 2.3 that

$V_h^{(n)} \ge e_h^{(n)}$

By the same way, we also prove that

$V_h^{(n)} \ge -e_h^{(n)}$. This implies that

$$\| U_h^{(n)} - u_h(t_n)\|_\infty \le e^{(P+1)t_n}(\| \varphi_h - u_h(0)\|_\infty + Mh^2 + M\Delta t_n)$$

Now let us show that N=J, suppose that $N < J$ If we replace n by N in the above inequality and from (16) we find that

$$1 \le \| U_h^{(N)} - u_h(t_N)\|_\infty \le e^{(P+1)t_N}(\| \varphi_h - u_h(0)\|_\infty + Mh^2 + M\Delta t_n).$$

Since the term on the right hand side of the second inequality goes to zero as h tends to zero, we deduce that $1 \le 0$ which is a contradiction and the proof is complete.

Now, we can give the main theorem of the section.

Theorem 5.2:

Suppose that the problem (1)-(3) has a solution u which blows up in a finite time T_0 *and* $u \in C^{4,2}([-1,1]\times[0,T_0))$. *Assume that the initial data at (8) satisfies*

$$\| \varphi_h - u_h(0)\|_\infty = o(1) \quad as\ h \to 0. \qquad (20)$$

Under the assumption of theorem 3.1, the problem $(4)-(6)$ *has a solution* $U_h^{(n)}$, *which blows up in a finite time* $T_h^{\Delta t}$ *and following relation holds* $\lim_{h\to 0} T_h^{\Delta t} = T_0$.

Proof: We know from Remark 3.1 that $\frac{\tau}{\tau'}$ is bounded. Let $\varepsilon > 0$, there exists a constant $R > 0$ such that

$$\frac{\tau}{f(x)} + \frac{\tau}{\tau'}\int_x^{+\infty}\frac{ds}{f(s)} < \frac{\varepsilon}{2} \text{ for } x \in [R, +\infty).$$

Since u blows up at the time T_0, there exists $T_1 \in (T_0 - \frac{\varepsilon}{2}, T_0)$ such that $\| u(.,t)\|_\infty \geq 2R$ for $t \in [T_1, T_0)$. Let $T_2 = \frac{T_1 + T_2}{2}$ and q a positive integer such that $t_q = \sum_{n=0}^{q-1}\Delta t_n \in [T_1, T_2]$ for h small enough. We have $\sup_{t\in[0,T_2]} \| u(.,t)\|_\infty < +\infty$. It follows from *theorem 4.1* that the problem $(4) - (6)$ has a solution $U_h^{(n)}$ which verifies $\| U_h^{(n)} - u_h(t_n)\|_\infty < R$ for $n \leq q$ which implies

$\| U_h^{(q)}\|_\infty \geq \| u_h(t_q)\|_\infty - \| U_h^{(q)} - u_h(t_q)\|_\infty \geq R$

From theorem 3.1 $U_h^{(n)}$ blows up at a finite time $T_h^{\Delta t}$.
It follows remark 3.1 and (20) that

$$|T_h^{\Delta t} - t_q| \leq \frac{\tau}{f(\| U_h^{(q)}\|_\infty)} + \frac{\tau}{\tau'}\int_{\| U_h^{(q)}\|_\infty}^{+\infty}\frac{ds}{f(s)} \leq \frac{\varepsilon}{2}$$

Because $\| U_h^{(q)}\|_\infty \geq R$, we deduce that

$|T_0 - T_h^{\Delta t}| \leq |T_0 - t_q| + |t_q - T_h^{\Delta t}| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon.$ Which leads us to the result?

VI. NUMERICAL EXPERIMENTS

In this section, we present some numerical approximations to the blow-up time of
$(1) - (3)$. We approximate the solution u of (1)-(3) by the solution $U_h^{(n)}$ of the following explicit scheme:

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \delta^2 U_i^{(n)} + f(U_i^{(n)}), \quad 1 \leq i \leq I$$

$U_0^{(n)} = 0$,

$U_i^{(0)} = \varphi_i \geq 0, \quad 0 \leq i \leq I,$

Where

$$\Delta t_n = \min\left\{\frac{h^2}{3}; \frac{\tau}{f(\| U_h^{(n)}\|_\infty)}\right\}$$

We also approximate the solution u of $(1) - (3)$ by the solution $U_h^{(n)}$ of the implicit scheme below

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \delta^2 U_i^{(n+1)} + f(U_i^{(n)}), \quad 1 \leq i \leq I$$

$U_0^{(n)} = 0,$

$U_i^{(0)} = \varphi_i \geq 0, \quad 0 \leq i \leq I$.

Where

$$\Delta t_n = \frac{\tau}{f(\| U_h^{(n)}\|_\infty)}, \quad \tau = h^2$$

In both cases, we take $\varphi_i = \varepsilon \sin(\frac{i\pi h}{2}), \ 1 \leq i \leq I$

The problem described may be written as follows

$$U_i^{(n+1)} = U_i^{(n)} + \Delta t_n \delta^2 U_i^{(n+1)} + \Delta t_n f(U_i^{(n)})$$

$$U_i^{(n+1)} = U_i^{(n)} + \Delta t_n \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} + \Delta t_n f(U_i^{(n)})$$

$$U_i^{(n+1)} = \left(1 - \frac{2\Delta t_n}{h^2}\right)U_i^{(n)} + \frac{\Delta t_n}{h^2}\left(U_{i+1}^{(n)} + U_{i-1}^{(n)}\right) + \Delta t_n f(U_i^{(n)})$$

$$U_i^{(n+1)} = \frac{\Delta t_n}{h^2}(U_{i+1}^{(n)}) + \left(1 - \frac{2\Delta t_n}{h^2}\right)U_i^{(n)} + \frac{\Delta t_n}{h^2}(U_{i-1}^{(n)}) + \Delta t_n f(U_i^{(n)})$$

$$For\ i = 1, U_1^{(n+1)} = \frac{\Delta t_n}{h^2}(U_2^{(n)}) + \left(1 - \frac{2\Delta t_n}{h^2}\right)U_1^{(n)} + \frac{\Delta t_n}{h^2}(U_0^{(n)}) + \Delta t_n f(U_1^{(n)})$$

$$For\ i = 1, \quad U_1^{(n+1)} = \left(1 - \frac{2\Delta t_n}{h^2}\right)U_1^{(n)} + \frac{\Delta t_n}{h^2}(U_2^{(n)}) + \Delta t_n f(U_1^{(n)})$$

$$For\ i = 2, \ U_2^{(n+1)} = \frac{\Delta t_n}{h^2}(U_1^{(n)})\left(1 - \frac{2\Delta t_n}{h^2}\right)U_2^{(n)} + \frac{\Delta t_n}{h^2}(U_3^{(n)}) + \Delta t_n f(U_2^{(n)})$$

Let us notice that the restriction on the time step $\Delta t_n \leq \frac{h^2}{3}$ ensure the nonnegativity of the discrete solution.

The implicit scheme may be written in the following form

$$\frac{U_i^{(n+1)}-U_i^{(n)}}{\Delta t_n}=\frac{U_{i+1}^{(n+1)}-2U_i^{(n+1)}+U_{i-1}^{(n+1)}}{h^2}+f(U_i^{(n)})U_0^{(n+1)}=0;$$

$$U_I^{(n+1)}=0$$

$$U_i^{(n+1)}=\frac{\Delta t_n}{h^2}U_i^{(n+1)}-\frac{2\Delta t_n}{h^2}U_i^{(n+1)}+\frac{\Delta t_n}{h^2}U_i^{(n+1)}+U_i^{(n)}+\Delta t_n f(U_i^{(n)})\left(1+\frac{2\Delta t_n}{h^2}\right)$$

$$U_i^{(n+1)}-\frac{\Delta t_n}{h^2}U_{i+1}^{(n+1)}-\frac{\Delta t_n}{h^2}U_{i-1}^{(n+1)}=U_i^{(n)}+\Delta t_n f(U_i^{(n)})$$

or

$$\frac{\Delta t_n}{h^2}U_{i-1}^{(n+1)}+\left(1+\frac{2\Delta t_n}{h^2}\right)U_i^{(n+1)}-\frac{\Delta t_n}{h^2}U_{i+1}^{(n+1)}=U_i^{(n)}+\Delta t_n f(U_i^{(n)})$$

$$For\ i=1,\ \left(1+\frac{2\Delta t_n}{h^2}\right)U_1^{(n+1)}-\frac{\Delta t_n}{h^2}U_2^{(n+1)}=U_1^{(n)}+\Delta t_n f(U_1^{(n)})$$

$$For\ i=2,\ -\frac{\Delta t_n}{h^2}U_1^{(n+1)}+\left(1+\frac{2\Delta t_n}{h^2}\right)U_2^{(n+1)}-\frac{\Delta t_n}{h^2}U_3^{(n+1)}=U_2^{(n)}+\Delta t_n f(U_2^{(n)})$$

$$\vdots$$

$$For\ i=I-1,-\frac{\Delta t_n}{h^2}U_2^{(n+1)}+\left(1+\frac{2\Delta t_n}{h^2}\right)U_{I-1}^{(n+1)}=U_1^{(n)}+\Delta t_n f(U_1^{(n)})$$

Then, we have

$A_h^{(n)}U_h^{(n+1)}=F_h^{(n)}$, where

$$A_h^{(n)}=\begin{pmatrix}1+2\frac{\Delta t_n}{h^2} & -\frac{\Delta t_n}{h^2} & 0 & \cdots & 0\\ -\frac{\Delta t_n}{h^2} & 1+2\frac{\Delta t_n}{h^2} & -\frac{\Delta t_n}{h^2} & 0 & \cdots\\ 0 & \ddots & \ddots & \ddots & \ddots\\ \vdots & \ddots & \ddots & \ddots & -\frac{\Delta t_n}{h^2}\\ 0 & \cdots & 0 & -\frac{\Delta t_n}{h^2} & 1+2\frac{\Delta t_n}{h^2}\end{pmatrix}$$

$$A_h^{(n)}=\begin{pmatrix}a_0 & b_0 & 0 & \cdots & 0\\ c_0 & a_0 & b_0 & 0 & \cdots\\ 0 & \ddots & \ddots & \ddots & \ddots\\ \vdots & \ddots & \ddots & \ddots & b_0\\ 0 & \cdots & 0 & c_0 & a_0\end{pmatrix}$$

With $a_0=1+2\frac{\Delta t_n}{h^2}$,

$b_0=-\frac{\Delta t_n}{h^2},\ i=1,...,I-2,$

$c_0=-\frac{\Delta t_n}{h^2},\quad i=1,...,I-1,$

$$\left(F^{(n)}\right)_i=U_i^{(n)}+\Delta t_n f(U_i^{(n)})$$

The matrix $A_h^{(n)}$ a three-diagonal matrix verifying the following properties:

$$\left(A_h^{(n)}\right)_{i,i}=\left(1+\frac{2\Delta t_n}{h^2}\right)>0,\quad 0\le i\le I$$

$$\left(A_h^{(n)}\right)_{i-1,i}=-\frac{\Delta t_n}{h^2}=\left(A_h^{(n)}\right)_{i,i+1}\le 0,\quad 2\le i\le I-2$$

$$\left(A_h^{(n)}\right)_{i,i}\ge\sum_{i\ne j}|\left(A_h^{(n)}\right)_{i,j}|$$

It follows that $U_h^{(n)}$ exists for $n\ge 0$. In addition, since $U_h^{(0)}$ is nonnegative, $U_h^{(n)}$ is also nonnegative for $n\ge 0$. We need the following definition.

Definition 6.1

We say that the discrete solution $U_h^{(n)}$ of the explicit or the implicit scheme blows up in a finite time if

$\lim_{n\to+\infty}\| U_h^{(n)}\|_\infty=+\infty$ and the series $\sum_{n=0}^{+\infty}\Delta t_n$ converges. The quantity $\sum_{n=0}^{+\infty}\Delta t_n$ is called the numerical blow-up time of the solution $U_h^{(n)}$.

In the following tables, in rows, we present the numerical blow-up time, values of n the CPU time and the orders of the approximations corresponding to meshes of 16, 32, 64, 128, 256, 512.

For the numerical blow-up time, we take

$t_n=\sum_{j=0}^{n-1}\Delta t_j$. Which is computed at the first time when

$\Delta t_n=|T_{n+1}-T_n|\le 10^{-16}$

The initial condition is $\varphi_i=\varepsilon\sin(\frac{i\pi h}{2}),\quad 1\le i\le I$

The order (S) of the method is computed from

$$s=\frac{\log((T_{4h}-T_{2h})/(T_{2h}-T_h))}{\log(2)}$$

The numerical experiments for $f(u)=\beta e^u$

First case: $\beta=10,\quad \varepsilon=0$

Table 1: *Numerical blow-up time, numbers of iterations, CPU time (seconds) and the approximations obtained with the implicit Euler method.*

I	t_n	n	CPUt	S
16	0.111327	842	1	-
32	0.110838	3228	5	-
64	0.111079	12350	24	1.57
128	0.111088	13956	33	1.58

Table 2: Numerical blow-up time, numbers of iterations, CPU time (seconds) and the approximations obtained with the explicit Euler method.

I	t_n	n	CPUt	S
16	0.111328	866	1	-
32	0.110848	3235	6	-
64	0.111149	12402	54	1.57
128	0.111188	13989	38	1.58

Second case: $\beta = 20\,,\ \varepsilon = 0$

Table 3: Numerical blow-up time, numbers of iterations, CPU time (seconds) and the approximations obtained with the implicit Euler method.

I	t_n	n	CPUt	S
16	0.052386	418	1	-
32	0.051354	1596	5	-
64	0.051378	6096	24	2.00
128	0.023249	23249	120	1.50

Table 4: Numerical blow-up time, numbers of iterations, CPU time (seconds) and the approximations obtained with the explicit Euler method.

I	t_n	n	CPUt	S
16	0.052389	454	1	-
32	0.051355	1598	6	-
64	0.051377	6194	26	2.00
128	0.051115	23255	124	1.50

Third case: $\boldsymbol{\beta = 100\,,\ \varepsilon = 0}$

Table 5: Numerical blow-up time, numbers of iterations, CPU time (seconds) and the approximations obtained with the implicit Euler method.

I	t_n	n	CPUt	S
16	0.011277	84	1	-
32	0.010322	319	3	-
64	0.010080	1216	23	1.70
128	0.001025	3708	30	2.00

Table 6: Numerical blow-up time, numbers of iterations, CPU time (seconds) and the approximations obtained with the explicit Euler method.

I	t_n	n	CPUt	S
16	0.011277	84	1	-
32	0.010322	319	3	-
64	0.010080	1216	23	1.70
128	0.001020	4633	30	2.00

Fourth case: $\boldsymbol{\beta = 5\,,\ \varphi_i = \sin(\frac{i\pi h}{2})}$

Table 7: Numerical blow-up time, numbers of iterations, CPU time (seconds) and the approximations obtained with the implicit Euler method.

I	t_n	n	CPUt	S
16	0.011277	84	1	-
32	0.010322	319	3	-
64	0.010080	1216	23	1.70
128	0.001025	3708	30	2.00

Table 8: Numerical blow-up time, numbers of iterations, CPU time (seconds) and the approximations obtained with the explicit Euler method.

I	t_n	n	CPUt	S
16	0.200514	1698	2	-
32	0.206628	6502	6	-
64	0.211201	24939	21	1.50
128	0211215	33708	45	2.00

Remark 6.1

In the case where the initial data is null, $\varphi = 0$ and the reaction term increases as a function of β, it is not hard to see that the blow-up time of the solution equals $\frac{1}{\beta}$. We observe from tables 1-6 that the numerical blow-up time tends to $\frac{1}{\beta}$ for $\beta = 10,\ \lambda = 20$ and $\beta = 100$.

When $\varphi_i = \sin\left(\frac{i\pi h}{2}\right)$ with $\beta = 5,$ it is not hard to see that the blow-up time of the solution equals$\frac{1}{\beta}$. See the tables 7-8

In the following, we also give some plots to illustrate our analysis. In the figures 1 to 6, we can appreciate that the discrete solution blows up globally. Let us notice that, theorecally, we know that the continuous solution blows up globally under the assumptions given in the introduction of the present paper (see [17], [46*])*

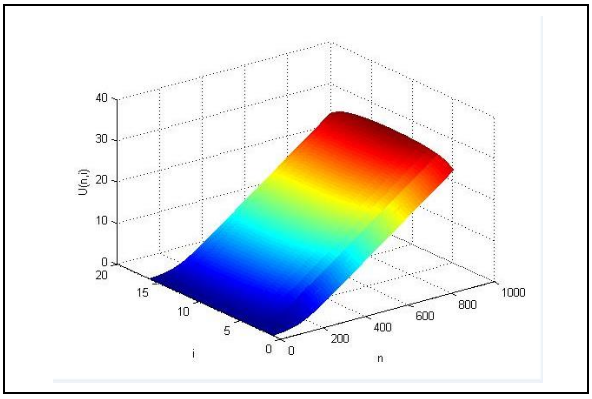

Fig1: Evolution of the discrete solution, source $f(u) = \beta e^{u}$ $\beta = 10, \varepsilon = 0$ *l* = 16 (Implicit scheme).

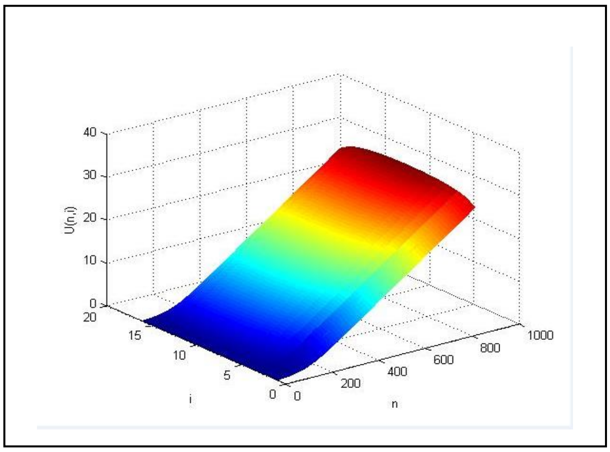

Fig2: Evolution of the discrete solution, source $f(u) = \beta e^{u}$ $\beta = 10, \varepsilon = 0$, *l* = 16 (explicit scheme).

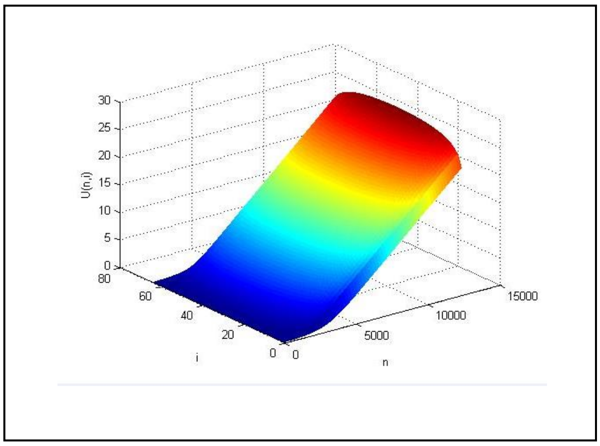

***Fig3:** Evolution of the discrete solution, Source $f(u) = \beta e^{u}$ $\beta = 10, \varepsilon = 0$; l = 64 (implicit scheme).*

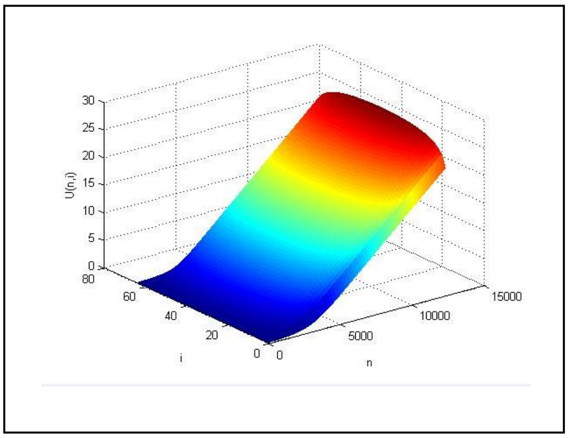

***Fig4:** Evolution of the discrete solution, Source $f(u) = \beta e^{u}$ $\beta = 10, \varepsilon = 0$; l = 64 (explicit scheme)*

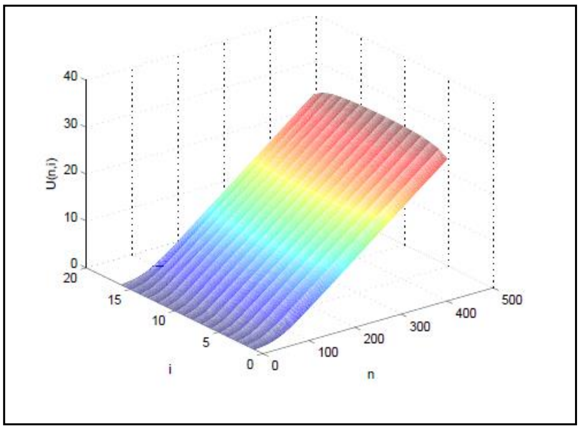

***Fig5:** Evolution of the discrete solution, Source $f(u) = \beta e^{u}$ $\beta = 20, \varepsilon = 0$; l = 64 (implicit scheme).*

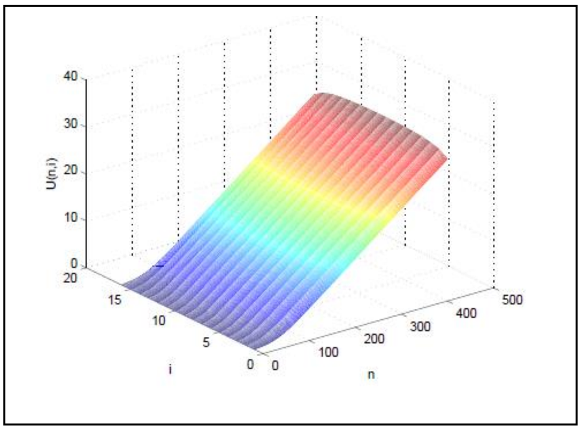

***Fig6:** Evolution of the discrete solution, Source $f(u) = \beta e^{u}$ $\beta = 20, \varepsilon = 0$; l = 64 (explicit scheme).*

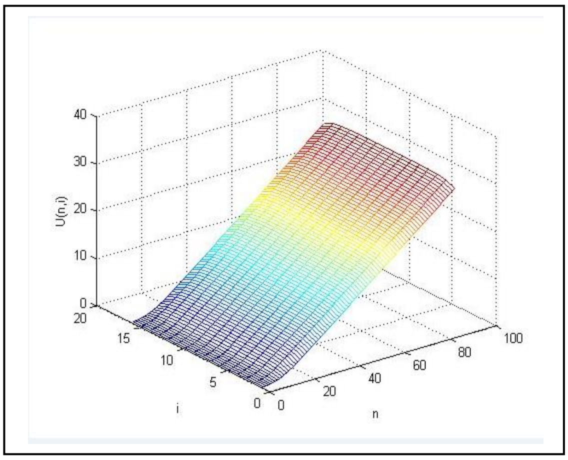

Fig7: *Evolution of the discrete solution,*
Source $f(u) = \beta e^u \quad \beta = 100, \varepsilon = 0$;
I = 16 (implicit scheme).

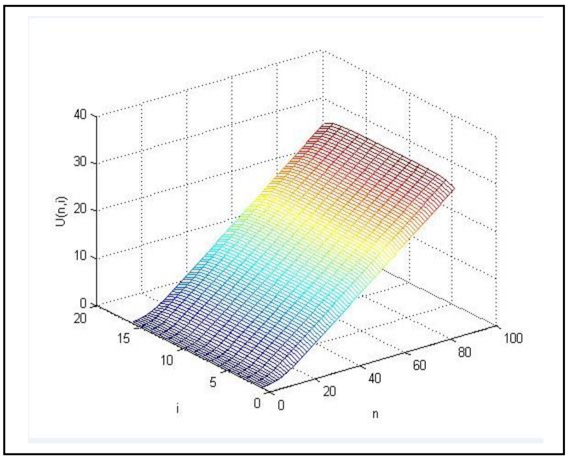

Fig8: *Evolution of the discrete solution,*
Source $f(u) = \beta e^u \quad \beta = 100, \varepsilon = 0$;
I = 16 (implicit scheme).

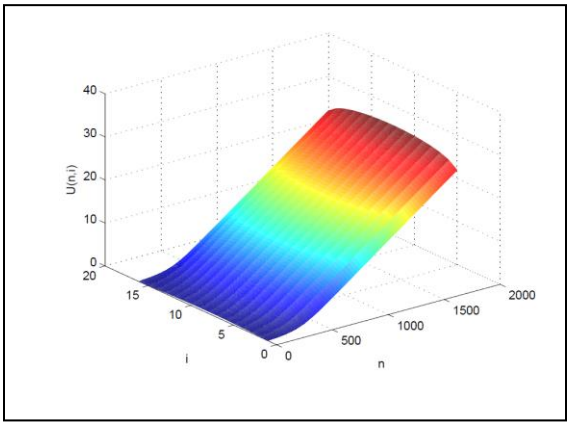

Fig9: *Evolution of the discrete solution,*
Source $f(u) = \beta e^u \quad \beta = 5, \varepsilon = 1$;
I = 16 (Implicit scheme).

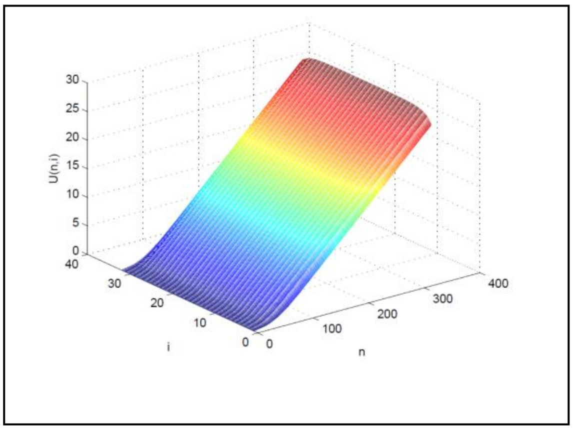

Fig 10: *Evolution of the discrete solution,*
Source $f(u) = \beta e^u \quad \beta = 100, \varepsilon = 0$;
I = 32 (implicit scheme).

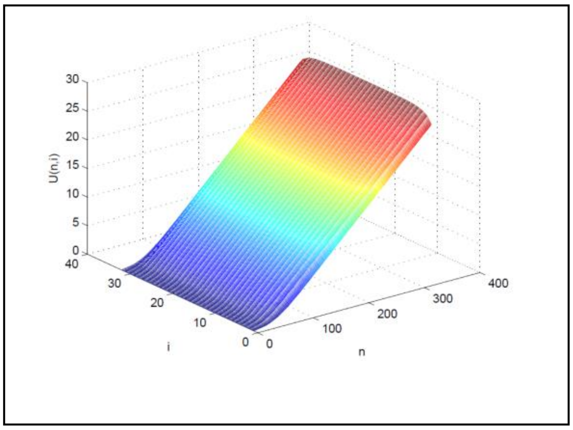

Fig11: *Evolution of the discrete solution,*
Source $f(u) = \beta e^u \quad \beta = 100, \varepsilon = 0$;
I = 32 (explicit scheme).

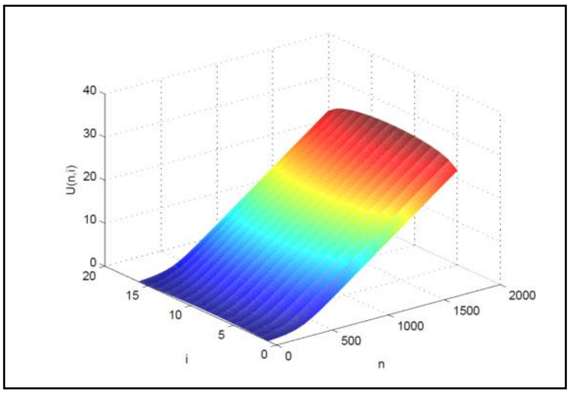

Fig12: *Evolution of the discrete solution,*
Source $f(u) = \beta e^u \quad \beta = 5, \varepsilon = 1$;
I = 16 (explicit scheme).

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