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# The Blow-Up Time For Reaction-Diffusion Equations With Dirichlet Boundary Conditions

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***Abstract -*** This paper is concerned with the study of the numerical approximation of the following initial-boundary value problem:

$$\begin{cases} u_t - u_{xx} - \dfrac{a}{x} u_x = f(u(x,t)), 0<x<1, t\in(0,T), \\ u(0,t)=0, t\in(0,T) \\ u(1,t)=0, t\in(0,T) \\ u(x,0)=u_0(x), 0\le x\le 1 \end{cases}$$

Where $a\ge 0$ , and $f(u)=u^q$ , where $q>0$. We obtain some conditions under which the positive solution of a semi-discrete form of the above problem blows up in a finite time and estimate its semi-discrete blow up time. Under some assumptions, we also show that the semi-discrete blow up time converges to the real one when the mesh parameter goes to zero. Finally, we give some numerical results to illustrate our analysis.

***AMS subject classification*** (2000): *35B40, 35B50, 35K60, 65M06*.
***Key-words***: *Semi-discretisations, discretisations, semi-linear heat equations, semi-discrete blow-up time.*

## I. Introduction

Consider the following initial boundary value problem:

$$u_t - u_{xx} - \frac{a}{x} u_x = f(u(x,t)), 0<x<1, t\in(0;T) (1)$$
$$u(0,t)=0, t\in(0,T) (2)$$
$$u(1,t)=0, t\in(0,T) \quad (3)$$
$$u(x,0)=u_0(x), 0\le x\le 1 (4)$$

Which models the temperature distribution of a large number of physical phenomenon from physics, chemistry and biology. It is known that a lot of physical and biological phenomena can be described by the following reaction-diffusion equation:

$$u_t(t,x)=u_{xx}(t,x)+f(u(x,t)) Eq \quad 1$$

Where the variable $u(t,x)$ can be seen as the temperature in a chemistry reaction or the population density of a biological species **(see [21]).**
The nonlinearity of $f$ stands for a net heat source (the natural heat source substract the dissipative heat source), which satisfies some growth conditions: it becomes positive when $u$ is large enough, or is positive for all $u>0$ .

The second order derivative $u_{xx}$ represents the diffusion. In a chemical reaction, we know that the chemical reaction will generate heat and the high temperature will accelerate the chemical reaction, thus the temperature will become higher and higher. Therefore, if the initial temperature is high enough, then the temperature will likely become high in a finite time, which is called a blow-up phenomenon.
Similarly, in a biological species, the large population density will bring the high birth rate, on the oder hand; the high birth rate will accelerate the population growth. Hence, if the initial population density is large enough, then blow-up will happen too.

Motivated by such phenomena arising in various applied fields, the blow-up problem for Eq1 has been studied by many authors (cf. [13], [22] and references therein). For example, the classical papers ([15], [17] ) consider the Cauchy problem:

$$u_t=u_{xx}+au+u^p (x\in R, t>0), u(0,x)=u_0(x)\ge 0 (x\in R) Eq2$$
,
with $a=0$ and prove that, when $1<p\le 3$ , the nontrivial solution must blow-up in a finite time ; when $p>3$ , this problem has time-global solutions for some small initial data. In [2], [8], [23], the autors studied this problem with $a<0$ and gave trichotomy result: any solution either blows up in a finite time, or vanishes $(i.e., u\to 0 as t\to\infty)$ , or converges to an evenly decreasing positive steady state as $t\to\infty$ .

The author in [18] studied the corresponding steady stats problem on annulus with $a<0$ . Moreover in [26], the author studied the corresponding steady stats problem of $Eq2$ , with $a\geq 0$ and estabilshes some existence and nonexistence results. The theoretical study of blow-up solutions for semi linear parabolic equations has been the subject of investigations of many authors (see [3],[6],[13],[16],[20],[27] and the references cited therein). The authors have proved that under some assumptions, the solution of (1)-(4) blows up in a finite time and the blow-up time is estimated.

Let us notice that if we consider the semi linear equation

$u_t=u_{xx}+u^p, x\in B, t\in(0,T)$,

With boundary conditions

$u(x,t)=0, x\in S, t\in(0,T)$,

And initial data

$u(x,0)=u_0(x)\in \overline{B}$ ,

Where $B=\{x\in R^n:|x|<1\}$ , $S=\{x\in R^n:|x|=1\}$ , the radial symmetric solutions are solutions of (1)-(4) with $a=0$ and $f(u(x,t))=u^p, p=n-1$ .

In this paper, we are interested in the numerical approximations of the problem (1) – (4) using a semi-discrete form of (1) – (4). Here $(0,T)$ is the maximal time interval on which the solution exists. The time $T$ may be finite or infinite. When $T$ is infinite, we say that the solution $u$ exists globally. When $T$ is finite, the solution $u$ develops a singularity in a finite time, namely :

$\lim_{t\to T}\|u(x,t)\|_\infty=+\infty$ .

Where $\|u(x,t)\|_\infty=\max_{0\leq x\leq 1}|u(x,t)|$ . In this case, we say that the solution $u$ blows up in a finite time and the time $T$ is called the blow-up time of the solution $u$ .

## II. SEMIDISCRETE PROBLEM

Let $I$ be a positive integer and define the grid $x_i=ih, 0\leq i\leq I$ , where $h=1/I$ . Approximate the solution $u$ of (1) – (4) by the solution $U_h(t)=(U_0(t),U_1(t),\ldots,U_I(t))^T$

$+¿U_i(t)+U_i^q(t), 1\leq i\leq I-1, t\in(0,T_b^h)$

$$\frac{dU_i(t)}{dt}=\delta^2U_i(t)+\frac{a}{ih}\delta^{¿} \quad , \quad (5)$$

$U_0(t)=0$ , $t\in(0,T_b^h)$ , (6)

$U_I(t)=0$ , $t\in(0,T_b^h)$ , (7)

$U_i(0)=\varphi_i$ , $0\leq i\leq I$ , (8)

Where

$$\delta^2U_i(t)=\frac{U_{i+1}(t)-2U_i(t)+U_{i-1}(t)}{h^2}, 1\leq i\leq I-1$$

$$+¿U_i(t)=\frac{U_{i+1}(t)-U_i(t)}{h}, 1\leq i\leq I-1.$$

$\delta^{¿}$

Here, $(0,T_b^h)$ is the maximal time interval on which $\|U_h(t)\|_\infty$ , is finite, where $\|U_h(t)\|_\infty=¿0\leq i\leq I|U_i(t)|$.

When $T_b^h$ is finite, we say that the solution $U_h(t)$ blows-up in a finite time and the time $T_b^h$ is called the blow-up time of the solution $U_h(t)$ . We give some conditions under which the solution of (5)-(8) blows up in a finite time and estimate its semidiscrete blow-up time. We also show that the semidiscrete blow-up time converges to the theoretical one when the mesh size goes to zero. A similar study has been undertaken in [1],[10],[12] and [14]. In [1] and [14], the authors have considered the equation (1) for $a=0$ and $f(u(x,t))=u^q$ with Dirichlet boundary conditions and nonnegative initial data. Our work was also motived by the papers in [1], [12] and [14]. In [1], the authors have considered the problem (1)--(4) in the case where the parameter a=0, and the source is $u^q$ .They have proved that the solution of the semi discrete scheme (5)-(8) blows up in a finite time and its semi discrete blow-up time converges to real one when the mesh size goes to zero in the case where the initial data is symmetric and large enough.

In [19], the author has shown that the solution of a discrete form of the equation $u_t=u_{xx}+u^2$ with Dirichlet boundary condition and large initial data blows up in a finite time and the discrete blow-up time converges to the real one when the mesh parameter goes to zero. In [3], semi discrete and discrete schemes have been used to study the phenomenon of extinction (we say that a solution extincts in a finite time if it reaches the values zero in a finite time). Numerical methods for heat equations with nonlinear boundary conditions have been described in [12], and [14].

The paper is organized as follows. In the next section, we give some properties concerning our scheme. In the fourth section, under some conditions, we prove that the solution of the semidiscrete problem (5)-(8) blows up in a finite time and estimate its semidiscrete blow-up time. In the fifth section, we study the convergence of the semidiscrete blow-up time. Finally, in the last section we report on some numerical experiments using several discretisations of (1)-(4).

## III . PROPRIETE OF THE SEMIDISCRETE SCHEME

In this section, we give some results about the discrete maximum principle. The following lemma is a discrete form of the maximum principle.

***Lemma 3.1***

*Let* $c_h(t) \in C^0([0,T], R^{I+1})$ *and* $V_h(t) \in C^1([0,T], R^{I+1})$ *such that for* $t \in (0,T)$

$$+¿V_i(t) + c_i(t)V_i(t) \geq 0 \, 1 \leq i \leq I-1$$

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) - \frac{a}{ih}\delta^{¿} \quad ,$$

$t \in (0, T_b^h)$ , (9)

$V_0(t) \geq 0, V_I(t) \geq 0$ (10)

$V_i(0) \geq 0, 0 \leq i \leq I$ (11)

*Then we have* $V_i(t) \geq 0, 0 \leq i \leq I, t \in (0,T)$ .

***Proof***:

Let $T_0 < T$ and let $m = V_i(t)$ . Since for $i \in \{0, \ldots, I\}$ , $V_i(t)$ is a continuous function, there exists $t_0 \in [0, T_0]$ such that *m*= $V_{i_0}(t_0)$ for a certain $i_0 \in \{0, \ldots, I\}$ . If $i_0 = 0$ or $i_0 = I$ , we have $m > 0$ . When $i_0$ is between 1 and $I-1$ , we observe that :

$$\frac{dV_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{V_{i_0}(t_0) - V_{i_0}(t_0 - k)}{k} \leq 0 \, (12)$$

$$\delta^2 V_{i_0}(t_0) = \frac{V_{i_0+1}(t_0) - 2V_{i_0}(t_0) + V_{i_0-1}(t_0)}{h^2} \geq 0,$$

$$if \, 1 \leq i_0 \leq I-1 \, (13)$$

$$+¿V_{i_0}(t_0) = \frac{V_{i_0+1}(t_0) - V_{i_0}(t_0)}{h} \geq 0, if \, 1 \leq i_0 \leq I-1 \, (14)$$

$$\delta^{¿}$$

,

Define the vector $Z_h(t) = e^{\lambda t} V_h(t)$ where $\lambda$ is such that $c_{i_0}(t_0) - \lambda > 0$ .

A straightforward computation reveals that

$$(¿¿i_0(t_0) - \lambda) Z_{i_0}(t_0) \geq 0 \quad c$$

$$+¿Z_{i_0}(t_0) + ¿$$

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) - \frac{a}{i_0 h}\delta^{¿}$$

$if \, 1 \leq i_0 \leq I-1$ . (15)

We observe from (5)-(8) that

$$\frac{dZ_{i_0}(t_0)}{dt} \leq 0 \; , \; \delta^2 Z_{i_0}(t_0) \geq 0 \; , \; \text{and}$$

$$+¿Z_{i_0}(t_0) \geq 0 \; .$$

$$\delta^{¿}$$

Using (9), we arrive at $(¿¿i_0(t_0) - \lambda) Z_{i_0}(t_0) \geq 0$ , (with $c$ above and $¿$ below) which implies that $Z_{i_0}(t_0) \geq 0$

Therefore $V_{i_0}(t_0) = m \geq 0$ and we have the desired result, and the proof is complete.

Another version of the discrete maximum principle is the following comparison lemma.

***Lemma 3.2***

*Let* $V_h(t), U_h(t) \in C^1([0,T], R^{I+1})$ *and* $g \in C^0(R \times R, R)$ *such that for* $t \in (0,T)$

$$+¿U_i(t) + g(U_i(t), t), 1 \leq i \leq I-1 \, (16)$$

$$+¿V_i(t) + g(V_i(t), t) < \frac{dU_i(t)}{dt} - \delta^2 U_i(t) - \frac{a}{ih}\delta^{¿}$$

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) - \frac{a}{ih}\delta^{¿}$$

$V_0(t) < U_0(t), V_I(t) < U_I(t)$ , (17)

$V_i(0) < U_i(0)$ , $0 \leq i \leq I$ . (18)

*Then we have* $V_i(t) < U_i(t)$ , $t \in (0,T)$ .

***Proof*** :

Define the vector $Z_h(t) = U_h(t) - V_h(t)$ . Let $t_0$ be the first $t > 0$ such that $Z_h(t) > 0$ for $t \in (0, t_0)$ , but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \ldots, I\}$ . If $i_0 = 0$ or $i_0 = I$ , we have a contradiction because of (17).

If $i_0$ is between $0$ and $I-1$ , we observe that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \geq 0, if \, 1 \leq i \leq I-1$$

,

$$+¿Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - Z_{i_0}(t_0)}{h} \geq 0, if \, 1 \leq i \leq I-1 \quad ,$$

$$\delta^{¿}$$

Which implies that

$$+¿Z_{i_0}(t_0)+g(U_{i_0}(t_0),t_0)-g(V_{i_0}(t_0),t_0)\le 0$$

$$\frac{dZ_{i_0}(t_0)}{dt}-\delta^2 Z_{i_0}(t_0)-\frac{a}{ih}\delta^{¿}$$ , if

$1\le i\le I-1$

But this inequality contradict (16).This ends the proof.

## IV. SEMIDISCRETE BLOWING-UP SOLUTIONS

In this section under some assumptions, we show that the solution $U_h$ of the semidiscrete problem (5)-(8) blows up in a finite time and estimate its semidiscrete blow-up time. We need the following important lemma. This lemma at the first gives us a property of the operator $\delta^2$ , then reveals a property of the operator $\overset{+¿}{\delta^{¿}}$ .

***Lemma 4.1***

Let $U_h\in R^{I+1}$ such that $U_h\ge 0$ . Then we have

$$\delta^2 f(U_i)\ge f'(U_i)\delta^2 U_i \ , \quad 0\le i\le I$$

$$+¿U_i$$

$$+¿f(U_i)\ge f'(U_i)\delta^{¿} \ , \quad 0\le i\le I$$

$$\delta^{¿}$$

**Proof :**

We apply Taylor's expansion to obtain :

$$f(U_{i+1})=f(U_i)+(U_{i+1}-U_i)f'(U_i)+\frac{(U_{i+1}-U_i)^2}{2}f''(\theta$$

, for $0\le i\le I-1$

$$f(U_{i-1})=f(U_i)+(U_{i-1}-U_i)f'(U_i)+\frac{(U_{i-1}-U_i)^2}{2}f''($$

, for $1\le i\le I$

where $\theta_i$ is an intermediate values between $U_i$ and $U_{i+1}$ , and $\eta_i$ is an intermediate value between $U_{i-1}$ and $U_i$ .

These two equations above imply that

$$f(U_{i+1})-2f(U_i)+f(U_{i-1})=(U_{i+1}-2U_i+U_{i-1})f'(U_i)\cdot$$

.

The first equation and the last one imply that :

$$\delta^2 f(U_i)=f'(U_i)\delta^2 U_i+\frac{(U_{i+1}-U_i)^2}{2h^2}f''(\theta_i)+\frac{(U_{i+1}-U_i}{2h^2}$$

.

$$+¿U_i+\frac{(U_{i+1}-U_i)^2}{2h}f''(\theta_i)$$

$$+¿f(U_i)=f'(U_i)\delta^{¿}$$ .

$$\delta^{¿}$$

Using the convexity of $f$ we obtain the desired result.

***Lemma: 4.2***

*Let $U_h$ be the solution of (5)-(8). Then we have*

$$U_{i+1}(t)>U_i(t),0\le i\le I-1,t\in(0,T_b^h),(19)$$

**Proof.**

Let $t_0$ be the first $t>0$ , such that $U_{i+1}(t)-U_i(t)>0$ for $0\le i\le I-1$ , but $U_{i_0+1}(t)=U_{i_0}(t)$ for a certain $i_0\in\{0,\dots,I-1\}$ . Without lost of generality we may suppose that $i_0$ is the smallest integer which satisfies the above equality.

Now let us introduce the functions $Z_i(t)=U_{i+1}(t)-U_i(t)$ for $0\le i\le I-1$ , $t>0$ .

We get

$$\frac{dZ_{i_0}(t_0)}{dt}=\lim_{k\to 0}\frac{Z_{i_0}(t_0)-Z_{i_0}(t_0-k)}{k}\le 0$$

$$\delta^2 Z_{i_0}(t_0)=\frac{Z_{i_0+1}(t_0)-2Z_{i_0}(t_0)+Z_{i_0-1}(t_0)}{h^2}\ge 0, if\ 1\le i\le I-1$$

,

$$+¿Z_{i_0}(t_0)=\frac{Z_{i_0+1}(t_0)-Z_{i_0}(t_0)}{h}\ge 0, if\ 0\le i\le I-1$$ .

$$\delta^{¿}$$

which implies that

$$+¿Z_{i_0}(t_0)+f(U_{i_0+1}(t_0),t_0)-f(U_{i_0}(t_0),t_0)\le 0$$

$$\frac{dZ_{i_0}(t_0)}{dt_0}-\delta^2 Z_{i_0}(t_0)-\frac{a}{i_0 h}\delta^{¿}$$ , if

$0\le i_0\le I-1$

But this inequality contradicts (5)-(8) and we obtain the desired result.

***Theorem 4.1***

*Suppose that there exists a positive integer $A$ such that the initial data at (8) satisfies*

$$+¿U_i(0)+U_i^q(0)\ge AU_i^q(0)$$

$$\delta^2 U_i(0)+\frac{a}{ih}\delta^{¿} \ , \quad 0\le i\le I\ (20)$$

*Then the solution $U_i(t)$ of (5)-(8) blows up in a finite time $T_b^h$ and we have the following estimate*

$$T_b^h\le\frac{1}{A}\frac{\|U_h(0)\|_\infty^{1-q}}{(q-1)}$$

***Proof***

Since $(0,T_b^h)$ be the maximal time interval on which $\|U_h(t)\|_\infty$ is finite, our aim is to show that $T_b^h$ is finite and obeys the above inequality. Introduce the following functions $f(U)=U^q$ and $J_h$ such that :

$J_i = \frac{dU_i}{dt} - AU_i^q, 0 \le i \le I.$

we shall prove that $J_h \ge 0, t \ge 0, 0 \le i \le I$ .

By a straightforward computation we get:

$$\frac{dJ_i}{dt} - \delta^2 J_i = \frac{d}{dt}\left(\frac{dU_i}{dt} - \delta^2 U_i\right) - AqU_i^{q-1}\frac{dU_i}{dt}$$

$+A\delta^2 U_i^q, 0 \le i \le I$ .

From *Lemma 4.1*

$\delta^2 f(U_i) \ge f'(U_i)\delta^2 U_i$ , $0 \le i \le I$

$+¿ U_i$

$+¿ f(U_i) \ge f'(U_i)\delta^{¿}$ , $0 \le i \le I$

$\delta^{¿}$

we have $\delta^2 U_i^q \ge qU_i^{q-1}\delta^2 U_i$ which implies that

$$\frac{dJ_i}{dt} - \delta^2 J_i \ge \frac{d}{dt}\left(\frac{dU_i}{dt} - \delta^2 U_i\right) - AqU_i^{q-1}\left(\frac{dU_i}{dt} - \delta^2 U_i\right.$$

Use (5)-(8) to obtain

$+¿ U_i$

$+¿ \frac{dU_i}{dt} - AqU_i^{q-1}\delta^{¿}$

$\delta^{¿},$

$\frac{dJ_i}{dt} - \delta^2 J_i \ge qU_i^{q-1}J_i + \frac{a}{ih}¿$

$0 \le i \le I.$ (21)

Taking into account the expression of $J_h(t)$ , we get

$q$

$A\delta_i^{¿}$

$+¿ \frac{dU_i}{dt} - ¿$ , which implies that

$+¿ J_i = \delta^{¿}$

$\delta^{¿}$

$q$

$A\delta_i^{¿}$

$+¿ J_i + ¿$

$+¿ \frac{dU_i}{dt} = \delta^{¿}$ . From *Lemma 4.1*, we arrive at

$\delta^{¿}$

$+¿ U_i$

$+¿ J_i + AU_i^{q-1}\delta^{¿}$ . Therefore, we get

$\frac{dU_i}{dt} \ge \delta^{¿}$

$+¿ J_i$

$+¿ U_i \ge \delta^{¿}$

$\frac{dU_i}{dt} - AU_i^{q-1}\delta^{¿}$ band due to (20), we discover that

$+¿ \frac{dJ_i}{dt} \ge qU_i^{q-1}J_i, 0 \le i \le I$ . Obviously,

$\frac{dJ_i}{dt} - \delta^2 J_i - \frac{a}{ih}\delta^{¿}$

$J_0(t) = 0, J_I(t) = 0$ and the hypotheses on the initial data ensure that $J_h(0) \ge 0$ . It follows from *Lemma 3.1* that $J_h(t) \ge 0$ for $t \in (0, T_b^h)$ , which implies that

$\frac{dU_i}{dt} \ge AU_i^q$ , $0 \le i \le I$ . This estimation may be rewritten as follows :

$$\frac{dU_i}{U_i^q} \ge Adt \quad , \quad 0 \le i \le I \qquad (22)$$

Integrating this inequality over $(0, T_b^h)$ , we find that :

$$T_b^h - t \le \frac{1}{A}\frac{U_i^{1-q}(t)}{(q-1)} \quad , \quad 0 \le i \le I$$

(23)

which implies that $T_b^h \le \frac{1}{A}\frac{\|U_h(0)\|_\infty^{1-q}}{(q-1)}$ . Therefore $T_b^h$ is finite and the proof is complete.

***Remark 4.1***

*The inequalities (22) imply that*

$T_b^h - t_0 \le \frac{1}{A}\frac{\|U_h(t_0)\|_\infty^{1-q}}{(q-1)}$ , *if* $0 \le t_0 \le T_b^h$ *and there exists a constant* $B > 0$ *such that*

$$U_i(t) \le \frac{B}{(T_b^h - t)^{\frac{1}{q-1}}}, 0 \le i \le I.$$

***Theorem 4.2***

*Let* $U_h(t)$ *be the solution of (5)-(8). Then we have*

$$T_b^h \ge \frac{\|U_h(0)\|_\infty^{1-q}}{(q-1)}.$$

***Proof:***

Let $i_0$ be such that $U_{i_0}(t) = \max_{0 \le i \le I} U_i(t)$ . It is not hard to see that

$$\delta^2 U_{i_0}(t) = \frac{U_{i_0+1}(t_0) - 2U_{i_0}(t_0) + U_{i_0-1}(t_0)}{h^2} \le 0, if\ 1 \le i \le I-1$$

$$+¿ U_{i_0}(t) = \frac{U_{i_0+1}(t) - U_{i_0}(t)}{2h} \le 0, if\ 0 \le i \le I-1.$$

$\delta^{¿}$

We deduce that $\frac{dU_{i_0}}{dt} \le U_{i_0}^q$ , which implies that

$\frac{dU_i}{U_i^q} \leq dt$ . Integrating this inequality over $(0, T_b^h)$ ,

we obtain $T_b^h \geq \frac{U_{i_0}^{1-q}(0)}{(q-1)}$ .

Use the fact that $U_{i_0}(0) = \|U_h(0)\|_\infty$ , we obtain

$T_b^h \geq \frac{\|U_h(0)\|_\infty^{1-q}}{(q-1)}$ and the theorem is completely proved.

## V. CONVERGENCE OF THE SEMIDISCRETE BLOW-UP TIME

In this section, we prove the convergence of the semidiscrete blow-up time. Under some assumptions, we show that the blow-up time of the solution for the semidiscrete problem converges to the real one when the mesh size tends to zero. In order to prove this result, firstly we show that for each fixed time interval $[0, T]$ where the solution $u$ of (1)-(4) is defined, the solution $U_h(t)$ of (5)-(8) approximates $u$ , when the mesh parameter $h$ goes to zero by the following theorem.

***Theorem 5.1***

*Assume that (1)-(4) has a solution* $u \in C^{4,1}([0,1] \times [0,T])$ *and the initial condition at (8) satisfies*

$\|\varphi_h - u_h(0)\|_\infty = o(1)$ *as* $h \to 0$

*(24)*

*where* $u_h(t) = (u(x_0, t), \ldots, u(x_I, t))^T$ *Then, for* $h$ *sufficiently small, the problem (5)-(8) has a unique solution* $U_h(t) \in C^1([0,T], R^{I+1})$ *such that*

$\max_{0 \leq t \leq T} \|U_h(t) - u_h(t)\|_\infty = o(\|\varphi_h - u_h(0)\|_\infty + h)$ *as* $h \to 0$ *(25)*

**Proof**

Since $u \in C^{4,1}$ , there exist positive constants $K$ and $M$ such that

$\|u\|_\infty \leq K, q(K+1)^{q-1} \leq M$ (26)

The problem (5)-(8) has for each $h$ a unique solution $U_h(t) \in C^1([0, T_b^h], R^{I+1})$ . Let $t(h)$ the greatest value of $t > 0$ such that

$\|U_h(t) - u_h(t)\|_\infty < 1$ for $t \in (0, t(h))$ .

(27)

The relation (24) implies that $t(h) > 0$ for $h$ sufficiently small. Let $t^{¿}(h) = min(t(h), T)$ .

By the triangle inequality, we obtain

$\|U_h(t)\|_\infty \leq \|u_h(t)\|_\infty + \|U_h(t) - u_h(t)\|_\infty$ ,for $t \in (0, t^{¿}(h))$ , which implies that

$\|U_h(t)\|_\infty \leq 1 + K$ for $t \in (0, t^{¿}(h))$ .

(28)

Let $e_h = U_h(t) - u_h(t)$ be the error of discretization.

Since $u \in C^{4,1}$ , using Taylor's expansion, we have for $t \in (0, t^{¿}(h))$ ,

$+¿ e_i(t) = q\xi_i^{q-1} e_i + o(h)$

$\frac{de_i(t)}{dt} - \delta^2 e_i(t) - \frac{a}{ih}\delta^{¿}$ ,

where $\xi_i$ is an intermediate value between $U_i(t)$ and $u(x_i, t)$ . Using (26) and (28), there exists a positive constant $M$ such that

$+¿ e_i(t) \leq M|e_i(t)| + Mh$

$\frac{de_i(t)}{dt} - \delta^2 e_i(t) - \frac{a}{ih}\delta^{¿}$ ,

$0 \leq i \leq I$ . (29)

Introduce the vector $Z_h(t)$ such that

$Z_i(t) = e^{(M+1)}(\|\varphi_h - u_h(0)\|_\infty + Mh)$ , $0 \leq i \leq I$ .

A straightforward calculation reveals that

$+¿ Z_i(t) > MZ_i + Mh$

$\frac{dZ_i(t)}{dt} - \delta^2 Z_i(t) - \frac{a}{ih}\delta^{¿}$ , $0 \leq i \leq I$ ,

$Z_I(t) > e_I(t)$ .

It follows from Lemma 3.2 that $Z_h(t) > e_h(t)$ for $t \in (0, t^{¿}(h))$ .

In the same way, we also prove that $Z_h(t) > -e_h(t)$ for $t \in (0, t^{¿}(h))$ , which implies that

$\|U_h(t) - u_h(t)\|_\infty \leq e^{(M+1)}(\|\varphi_h - u_h(0)\|_\infty + Mh)$

$0 \leq i \leq I$ .

Let us show that $t^{¿}(h) = T$. Suppose that $T > t(h)$. From (27), we obtain

$1 = \|U_h((th)) - u_h(t(h))\|_\infty \leq e^{(M+1)}(\|\varphi_h - u_h(0)\|_\infty + Mh)$

. (30)

Since the term on the right hand side of the above inequality $1 \leq 0$ is impossible.

Consequently $t^{¿}(h) = T$ , and the proof is complete.

Now, we are in a position to prove the main theorem of this section.

***Theorem 5.2***

Suppose that the problem (1)-(4) has a solution $u$ which blows up in a finite time $T_b$ such that $u \in C^{4,1}([0,1] \times [0,T])$ and the initial data at (4) satisfies $\|\varphi_h - u_h(0)\|_\infty = o(1)$ *as* $h \to 0$ .

Under the assumptions of ***Theorem 4.1***, the problem (5)-(8) has a solution $U_h(t)$ which blows upin a finite time $T_b^h$ and $\lim_{h\to 0} T_b^h = ¿ T_b ¿$ .

***Proof :***
Let $\varepsilon > 0$ . There exists a positive constant $N$ such that

$$\frac{1}{A}\frac{x^{1-q}}{(q-1)} \le \frac{\varepsilon}{2} < \infty \quad \text{for} \quad x \in ¿ N, +\infty ¿ . \qquad (31)$$

Since $u$ blows up at the time $T_b$ , then there exists $T_1$ such that $|T_1 - T_b| \le \frac{\varepsilon}{2}$ and $\|u(x,t)\|_\infty \ge 2N$ for $t \in [T_1, T_b]$ . Letting $T_2 = \frac{T_1 + T_b}{2}$ , we see that $¿_{t\in[0,T_2]} |u(x,t)| < \infty$ .
It follows from ***Theorem 5.1*** that the problem (5)-(8) has a solution $U_h(t)$ which obeys $¿_{t\in[0,T_2]} |U_h(t) - u_h(t)|_\infty \le N$ . Applying the triangle inequality, we get $\|U_h(t)\|_\infty \ge \|u_h(t)\|_\infty - \|U_h(t) - u_h(t)\|_\infty$, which leads to $\|U_h(t)\|_\infty \ge N$ for $t \in [0, T_2]$ . From ***Theorem 4.1***, $U_h(t)$ blows up at the time $T_b^h$ .
We deduce from Remark and (31) that

$$|T_b^h - T_b| \le |T_b^h - T_2| + |T_2 - T_b| \le \frac{\varepsilon}{2} + \frac{1}{A}\frac{\|U_h(T_2)\|_\infty^{1-q}}{(q-1)} \le$$

, and we have the desired result.

## VI NUMERICAL EXPERIMENTS

In this section, we study the phenomenon of blow-up, using full discrete schemes (explicit and implicit) of (1)-(4). Firstly, we approximate the solution $u$ of (1)-(4) by the solution $U_h^{(n)} = \left(U_0^{(n)}, U_1^{(n)}, \ldots, U_I^{(n)}\right)^T$ of the, following explicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \delta^2 U_i^{(n)} + \frac{a}{ih}\frac{U_{i+1}^{(n)} - U_i^{(n)}}{h} + \left(U_i^{(n)}\right)^q, 1 \le i \le I$$

(32)

$$U_0^{(n)} = 0 \quad , \quad U_I^{(n)} = 0 \quad .$$

(33)

$$U_i^{(0)} = 20\cos\left(\frac{i\pi h}{2}\right) \quad , \quad 0 \le i \le I ,$$

(34)
where $n \ge 0$ ,

$$\Delta t_n = min\left\{\frac{h^2}{2+a}, h^2 \left\|U_h^{(n)}\right\|_\infty^{1-q}\right\} \quad ,$$

$$\delta^2 U_i^{(n)} = \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} \quad , \quad 1 \le i \le I-1 \quad .$$

Let us notice that the restriction on the time step guarantees the positivity of the discrete solution.
Secondly, approximate the solution $u$ of (1)-(4) by the solution $U_h^{(n)}$ of the following implicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \delta^2 U_i^{(n+1)} + \frac{a}{ih}\frac{U_{i+1}^{(n+1)} - U_i^{(n+1)}}{h} + \left(U_i^{(n)}\right)^q \quad , \quad 1 \le i \le I \quad ,$$

(35)

$$U_0^{(n+1)} = 0 \quad , \quad U_I^{(n+1)} = 0 \quad .$$

(36)

$$U_i^{(0)} = 20\cos\left(\frac{i\pi h}{2}\right) \quad , \quad 0 \le i \le I ,$$

(37)
where $\Delta t_n = h^2 \left\|U_h^{(n)}\right\|_\infty^{1-q}$ . The above equations may be rewritten in the following form

$$\frac{-\Delta t_n}{h^2} U_{i-1}^{(n+1)} + \left(1 + \frac{2\Delta t_n}{h^2} + \frac{a\Delta t_n}{ih^2}\right) U_i^{(n+1)} - \left(\frac{\Delta t_n}{h^2} + \frac{a\Delta t_n}{ih^2}\right) U_{i+1}^{(n+1)} \quad , \quad 1 \le i \le I-1 \quad (38)$$

$$U_0^{(n+1)} = 0 \quad , \quad U_I^{(n+1)} = 0 \quad .$$

(39)

The equalities (38)-(39) lead us to the linear system below
$A_h^{(n)} V_h^{(n+1)} = F_h^{(n)}$ , where $A_h^{(n)}$ is a $I \times I$ tridiagonal matrix defined as follows

$$A_h^{(n)} = \begin{pmatrix} a_0 & b_0 & 0 & & & & \\ c_1 & a_1 & b_1 & \cdots & & 0 & \\ 0 & c_2 & a_2 & & & & \\ & \vdots & & \ddots & & \vdots & \\ & & & & a_{I-3} & b_{I-3} & 0 \\ & 0 & & \cdots & c_{I-2} & a_{I-2} & b_{I-2} \\ & & & & 0 & c_{I-1} & a_{I-1} \end{pmatrix}$$

with $a_0 = 1 + \frac{2\Delta t_n}{h^2}(1+a)$ ,

$$a_i = 1 + \frac{2\Delta t_n}{h^2}(1+a) + \frac{a\Delta t_n}{ih^2} \quad , \quad i = 1, \ldots, I-1 \quad ,$$

$$b_0 = \frac{-2\Delta t_n}{h^2}(1+a) \quad , \quad b_i = -\left(\frac{2\Delta t_n}{h^2} + \frac{a\Delta t_n}{ih^2}\right) \quad ,$$

$$i = 1, \ldots, I-2 \quad , \quad c_i = \frac{-\Delta t_n}{h^2} \quad , \quad i = 1, \ldots, I-1 \quad ,$$

$$F_i^{(n)} = U_i^{(n)} + \Delta t_n \left(U_i^{(n)}\right)^q , 0 \le i \le I-1 \quad ,$$

$V_h^{(n+1)} = \left(U_0^{(n+1)}, U_1^{(n+1)}, \ldots, U_{I-1}^{(n+1)}\right)^T$ . It is not hard to see that $\left(A_h^{(n)}\right)_{ii} \geq 0$ , $\left(A_h^{(n)}\right)_{ij} \leq 0, i \neq j$ , $\left(A_h^{(n)}\right)_{ii} > \sum_{i \neq j} \left|\left(A_h^{(n)}\right)_{ij}\right|$ .

These inequalities imply that the linear system has a unique solution for $n \neq 0$ and the discrete solution is nonnegative. For the proof, see for instance [3]. We need the following definition.

***Définition 5.1***

We say that the solution $U_h^{(n)}$ of (32)-(34) or (35)-(37) blows up in a finite time if $\lim_{n \to +\infty} \left|U_h^{(n)}\right| = +\infty$ and the series $\sum_{n=0}^{+\infty} \Delta t_n$ converges.

The quantity $\sum_{n=0}^{+\infty} \Delta t_n$ is called the numerical blow-up time of the solution $U_h^{(n)}$ .

In the tables 1, 2, 3, 4, 5, 6, 7 and 8, we present the numerical blow-up times, values of $n$ , the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128. We take for the numerical blow-up time $T^n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when $\Delta t_n = \left|T^{n+1} - T^n\right| \leq 10^{-16}$ . The order(s) of the method is computed from

$$s = \frac{\log\left(\frac{T_{4h} - T_{2h}}{T_{2h} - T_h}\right)}{\log(2)}$$

**Numerical experiments** for $U_i^{(0)} = 20\cos\left(\frac{i\pi h}{2}\right), q = 2$ ,

***First case:*** $a = 0$

**Table 1**: Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method

| I | $t_n$ | $n$ | $CPU\ time$ | $s$ |
|---|---|---|---|---|
| 16 | 0.056343 | 7191 | 16 | - |
| 32 | 0.056231 | 27359 | 69 | - |
| 64 | 0.056210 | 103908 | 560 | 2.31 |
| 128 | 0.055203 | 405086 | 7052 | 1.68 |

**Table 2**: Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the implicit Euler method

| I | $t_n$ | $n$ | $CPU\ time$ | $s$ |
|---|---|---|---|---|
| 16 | 0.056376 | 7191 | 20 | - |
| 32 | 0.056240 | 27360 | 113 | - |
| 64 | 0.056213 | 103908 | 1460 | 2.22 |
| 128 | 0.055207 | 406009 | 20746 | 1.85 |

***Second case:*** $a = 1$

**Table 3**: Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method.

| I | $t_n$ | $n$ | $CPU\ time$ | $s$ |
|---|---|---|---|---|
| 16 | 0.067710 | 7406 | 13 | - |
| 32 | 0.066986 | 28264 | 70 | - |
| 64 | 0.066527 | 107691 | 594 | 0.76 |
| 128 | 0.066300 | 409301 | 20746 | 1.02 |

**Table 4**: Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the implicit Euler method

| I | $t_n$ | $n$ | $CPU\ time$ | $s$ |
|---|---|---|---|---|
| 16 | 0.067802 | 7407 | 18 | - |
| 32 | 0.067008 | 28265 | 109 | - |
| 64 | 0.0665323 | 107692 | 1506 | 0.84 |
| 128 | 0.066303 | 409302 | 21000 | 1.05 |

***Third case:*** $a = 1.5$

**Table 5**: Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method

| I | $t_n$ | $n$ | $CPU\ time$ | $s$ |
|---|---|---|---|---|
| 16 | 0.076519 | 7555 | 13 | - |
| 32 | 0.075299 | 28888 | 69 | - |
| 64 | 0.074569 | 110314 | 590 | 0.85 |
| 128 | 0.074177 | 435971 | 8580 | 0.95 |

**Table 6**: Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the implicit Euler method

| I | $t_n$ | $n$ | $CPU\ time$ | $s$ |
|---|---|---|---|---|
| 16 | 0.076700 | 7555 | 11 | - |
| 32 | 0.075332 | 28888 | 59 | - |
| 64 | 0.074577 | 110001 | 499 | 0.85 |
| 128 | 0.074177 | 420284 | 7580 | 0.95 |

***Fourth case:*** $a = 2$

**Table 7**: Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method

| I | $t_n$ | $n$ | $CPU\ time$ | $s$ |
|---|---|---|---|---|
| 16 | 0.090365 | 7765 | 15 | - |
| 32 | 0.089372 | 29778 | 71 | - |
| 64 | 0.087243 | 114065 | 630 | 0.85 |
| 128 | 0.086658 | 435971 | 8580 | 0.95 |

**Table 8**: Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the implicit Euler method

| I | $t_n$ | $n$ | $CPU\ time$ | $s$ |
|---|---|---|---|---|
| 16 | 0.090575 | 7767 | 19 | - |
| 32 | 0.088423 | 29780 | 115 | - |
| 64 | 0.087226 | 114067 | 1718 | 0.85 |
| 128 | 0.086636 | 435972 | 23051 | 1.07 |

In the following, we also give some plots to illustrate our analysis. In Figures 1 to 8, we can appreciate that the discrete solution blows up in finite time.

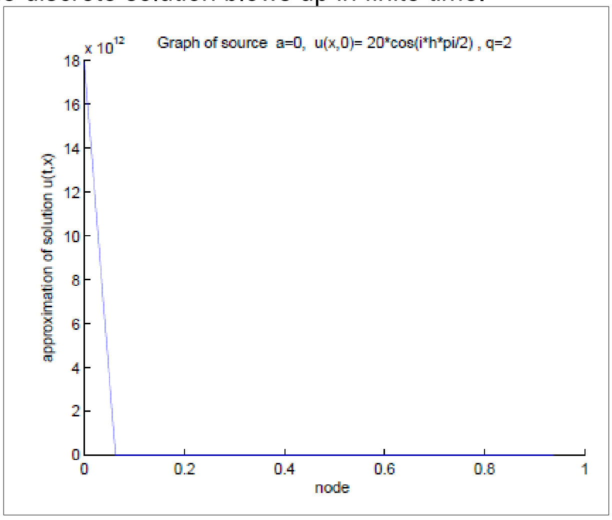


Fig1: Evolution of the discrete solution, source $f(u)=u^q\ q=2, a=0$ ; *I* = 16 (implicit scheme).

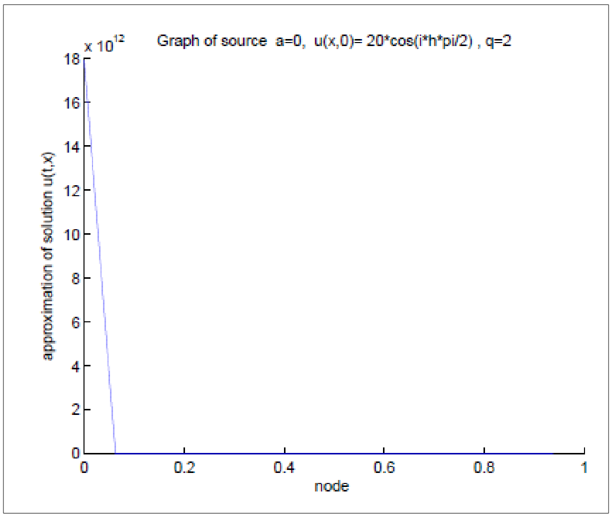


Fig2: Evolution of the discrete solution, source $f(u)=u^q\ q=2, a=0$ ; *I* = 16 (explicit scheme).

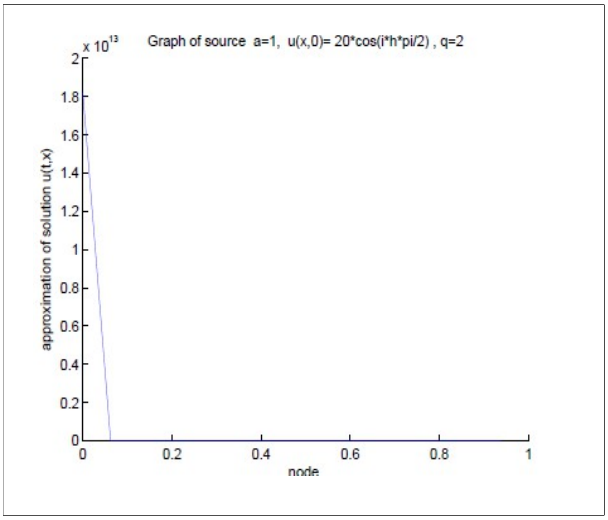


Fig3: Evolution of the discrete solution, source $f(u)=u^q\ q=2, a=1$ ; *I* = 16 (implicit scheme).

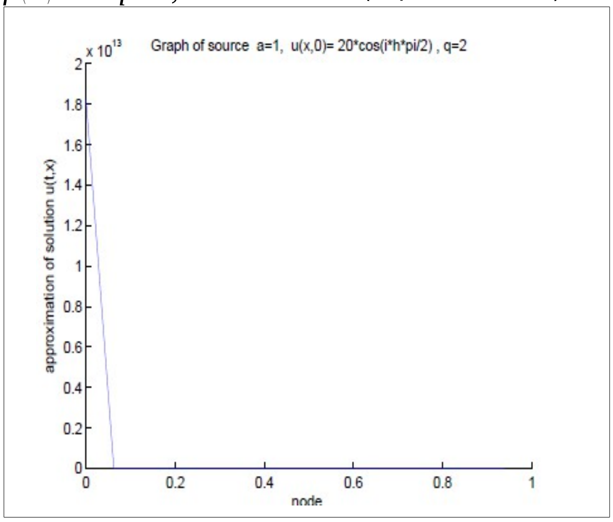


Fig4: Evolution of the discrete solution, source $f(u)=u^q\ q=2, a=1$ ; *I* = 16 (explicit scheme).

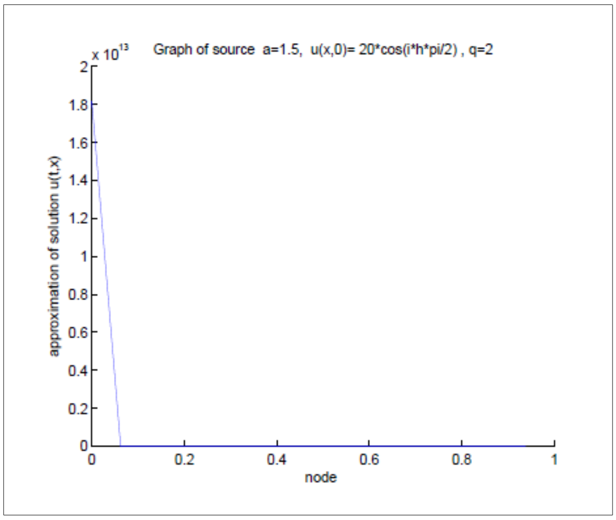


Fig5: Evolution of the discrete solution, source $f(u)=u^q \; q=2, a=1.5$ ; *I* = 16 (implicit scheme).

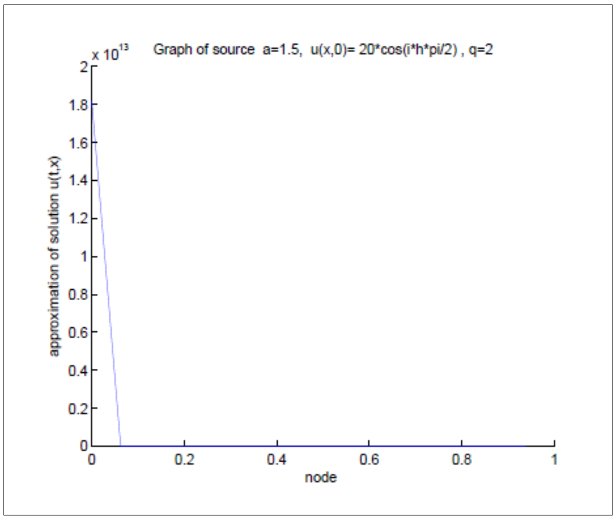


Fig6: Evolution of the discrete solution, source $f(u)=u^q \; q=2, a=1.5$ ; *I* = 16 (explicit scheme).

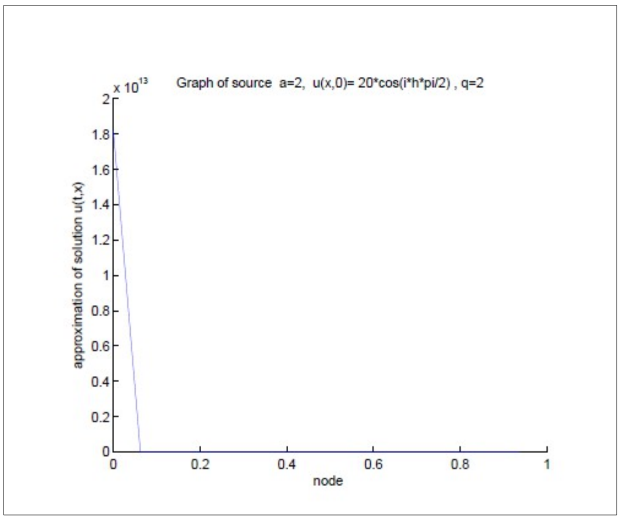


Fig7: Evolution of the discrete solution, source $f(u)=u^q \; q=2, a=2$ ; *I* = 16 (implicit scheme).

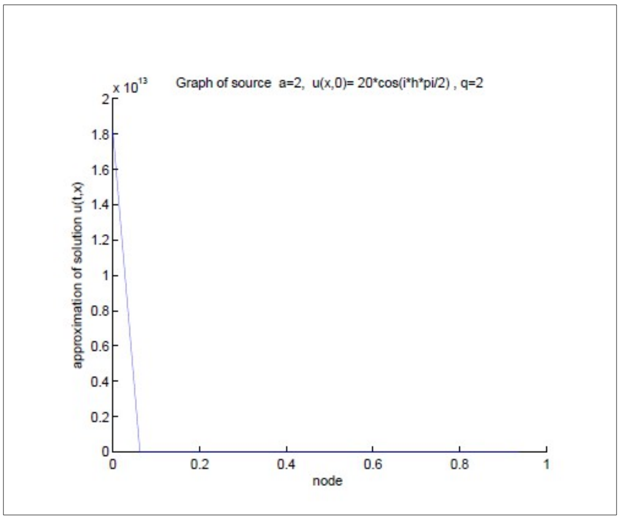


Fig8: Evolution of the discrete solution, source $f(u)=u^q \; q=2, a=2$ ; *I* = 16 (explicit scheme).

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