Set of values Schwartz derivative

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***Abstract*—The method solves the problem of parametric representations of the set Schwarz derivative values for classes S and S_M with an indication the boundary functions.**

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Introduction

This The main methods of geometric function theory are the ideas of parametric method, area method, variational method, method of integral representations. These methods have appeared in different times, and a cause for their design were different extreme problems, which include L. Bieberbach problem of coefficients holomorphic univalent in the unit circle on the class S, as well as the practical task of constructing conformal mappings of simply connected and multiply connected domains. In the paper the application of the method of parametric representations to the problem of finding the range of the Schwartz derivative for classes S and S_M. The use Globalized method to obtain a conformal mapping of one domain to another as a result of passage to the limit in the purpose-built family Shall maps in his first principle goes back to Charles Loewner [1]. About further development of the method see Ref. [2, 3].

The derivative of Schwartz

Definition and properties

Let f - the holomorphic univalent mapping of the domain $D\subset\mathbb{C}$. Then there are all derivatives $f^{(n)}(n=1,2,...)$, the holomorphic in D ,wherein $f'(z)\neq 0$ in D.

Schwartz derivative (or Schwartzman) function f is the a function

$$\Phi(z)=\frac{f'''(z)}{f'(z)}-\frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2,$$

Denoted by are also $\{f(z),z\},\{f,z\}$, that is $\{f(z),z\}=\{f,z\}=\Phi(z)$.

Note the following properties $\{f,z\}$.

1. Evidently, $\{z,z\}=0$.

2 .if $f(z)$ – linear fractional function, that is $f(z)=(az+b)/(cz+d)$, $ad-bc\neq 0$, that $\{f,z\}=0$.

3. We show that if $\zeta=F(w)$ – holomorphic function in $f(D)$,as well $w=f(z)$ – holomorphic function in D that

$$\{F(f(z)),z\}=\{F(w),w\}\big|_{w=f(z)}f'^2(z)+\{f(z),z\}. \qquad (1)$$

Indeed, since $[F(f(z))]'=F(f(z))f'(z)$, then we have a for the logarithmic derivative

$$\frac{[F(f(z))]''}{[F(f(z))]'}=\frac{F''(f(z))f'(z)}{F'(f(z))}+\frac{f''(z)}{f'(z)}. \qquad (2)$$

Therefore ,

$$\left(\frac{[F(f(z))]''}{[F(f(z))]'}\right)^2=\left(\frac{F''(f(z))f'(z)}{F'(f(z))}\right)^2+2\frac{F''(f(z))f'(z)}{F'(f(z))}+\left(\frac{f''(z)}{f'(z)}\right)^2. \qquad (3)$$

Let us differentiate (2) by z we get

$$\frac{[F(f(z))]'''}{[F(f(z))]'}-\left(\frac{[F(f(z))]''}{[F'(f(z))]'}\right)^2=\frac{F'''(f(z))(f'(z))^2}{F'(f(z))}+\frac{F''(f(z))f'(z)}{F'(f(z))}-$$

$$-\left(\frac{F''(f(z))f'(z)}{F'(f(z))}\right)^2+\frac{f'''(z)}{f'(z)}-\left(\frac{f''(z)}{f'(z)}\right)^2. \qquad (4)$$

Subtract from formula (4), formula of (3), preliminarily multiplied by $\frac{1}{2}$, we get (1).

Specifically, assuming $F(w)=(aw+b)/(cw+d)$, $ad-bc\neq 0$,we have $\{F(f(z)),z\}=\{f(z),z\}$

Integral representation of the Schwartz derivative

Let S class of holomorphic univalent in the circle $E=\{z\in\mathbb{C}:|z|<1\}$ function

$$f(z)=z+c_2z^2+c_3z^3+...$$

And S_M , $1\leq M<\infty$, – class limited at E functions f from the class S, that is, such that

$|f(z)|<M$ in E . consider $S_\infty=S$ .Class S_1 contains only one function $f(z)=z$.

The dense in sense of uniform convergence on compact subclass of the class S

(And S_M) functions $f(z;\mu)$ may be obtained from a population of solutions $\zeta=\zeta(\tau,z;\mu)$

Loewner equation

$$\frac{d\varsigma}{d\tau}=-\varsigma\frac{\mu(\tau)+\varsigma}{\mu(\tau)-\varsigma},0<\tau<M<\infty. \quad (5)$$

in which the $\mu(\tau)$ - a continuous function $|\mu(\tau)|=1$, satisfying the initial condition $\zeta(0,z;\mu)=z, z\in E$, according to the formula,

$$f(z;\mu)=\lim_{\tau\to\infty}e^{\tau}\zeta(\tau,z;\mu)$$

For Class S, and the formula

$$f(z;\mu)=M\zeta(\ln M,z;\mu)$$

For Class S_M. Functions $\zeta(\tau,z;\mu)=e^{-\tau}z+...$ univalent conformal display unit circle in the unit circle.

These statements are the basis of method of parametric representations

We apply this method to the determination of the set $\Delta(z_0)\subset$□ Schwarz derivative values $\Phi(z)=\{f,z\}$ at a fixed $z=z_0\in E$ on S and S_M Classes.

The set $\Delta(z_0)$ is limited, closed and connected. Imagine functionality $\{f,z_0\}$ integral.

Let $\zeta(\tau,z;\mu)$ – decision Loewner equation. Differentiating the identity

$$\frac{d\varsigma(\tau,z;\mu)}{d\tau}=-\varsigma(\tau,z;\mu)\frac{\mu(\tau)+\varsigma(\tau,z;\mu)}{\mu(\tau)-\varsigma(\tau,z;\mu)}$$

Here and below the prime denotes differentiation with respect to z.

Simple calculations give

$$\frac{d}{d\tau}\frac{\varsigma''(\tau,z;\mu)}{\varsigma'(\tau,z;\mu)}=-\frac{4\mu^2(\tau)\varsigma'(\tau,z;\mu)}{(\mu(\tau)-\varsigma(\tau,z;\mu))^3}. \quad (6)$$

And therefore

$$\frac{1}{2}\frac{d}{d\tau}\left(\frac{\varsigma''(\tau,z;\mu)}{\varsigma'(\tau,z;\mu)}\right)^2=-\frac{4\mu^2(\tau)\varsigma''(\tau,z;\mu)}{(\mu(\tau)-\varsigma(\tau,z;\mu))^3}. \quad (7)$$

Differentiating (6) by z and taking (7), we obtain

$$\frac{d}{d\tau}\{\varsigma(\tau,z;\mu),z\}=-\frac{12\mu^2(\tau)\varsigma'^2(\tau,z;\mu)}{(\mu(\tau)-\varsigma(\tau,z;\mu))^4}. \quad (8)$$

We have thus proved

Theorem 1.

Let $\mu(\tau)$, $0<\tau<M<\infty$, –continuous function and $\zeta(\tau,z;\mu)$. $\zeta(0,z;\mu)=z\in E$, – solution of the equation (5). Then the formula (8) takes place.

Consequence 1. The set of points

$$\{f(z_0,\mu),z_0\}=-12\int_0^{\infty}\frac{\mu^2(\tau)\varsigma'^2(\tau,z_0;\mu)}{(\mu(\tau)-\varsigma(\tau,z_0;\mu))^4}d\tau. \quad (9)$$

Dense in $\Delta(z_0)$ on the class S.

Consequence 2. The set of points

$$\{f(z_0,\mu),z_0\}=-12\int_0^{\ln M}\frac{\mu^2(\tau)\varsigma'^2(\tau,z_0;\mu)}{(\mu(\tau)-\varsigma(\tau,z_0;\mu))^4}d\tau$$

Dense in $\Delta(z_0)$ on the class S_M.

The set values of the functional $\Phi(z_0)$

in the class S

At point $z_0=0$ functional of $\Phi(z_0)=6(c_3-c_2{}^2)$.

Majorant region for the functional $\Phi(z_0)=6(c_3-c_2{}^2)=6J(0)$. To class S will get by following an assessment of the integral in (9) takes the form

$$J(0)=c_3-c_2{}^2=-2\int_0^{\infty}\frac{e^{-2\tau}}{\mu^2(\tau)}d\tau$$

Since $|\mu(\tau)|=1$, then

$$|J(0)|=|c_3-c_2{}^2|\le 2\int_0^{\infty}e^{-2\tau}d\tau=1$$

And consequently, $J(0)$ lies in the circle $|J|\le 1$. We will show that $J(0)$ – the closure of the unit circle, that is, that every point $\rho e^{i\theta}$, $0\le\rho\le 1$, $1\le\theta\le 2\pi$, is an value of the functional $c_3-c_2{}^2$ on the class S.

Function

$$K(z,\varphi)=\frac{z}{(1-e^{i\varphi}z)^2}=z+2e^{i\varphi}z^2+3e^{2i\varphi}z^3+...,$$

$$K_1(z,\varphi)=K(-z,\varphi),\ 1\le\varphi\le 2\pi,$$

Class S introduced into $J(0)$ point $e^{i\theta}=-e^{2i\varphi}$. For the region $K(E,\varphi)$ points

$$w(\lambda)=-\frac{e^{i\varphi}}{4\sin^2\frac{\lambda}{2}},\ 0\le\lambda\le\pi,$$

Are boundary points. It is easy to see that

$$L(z,\varphi,\lambda)=\frac{w(\lambda)K(z,\varphi)}{w(\lambda)-K(z,\varphi)}=\frac{z}{1-2\cos\lambda e^{i\varphi}z+e^{2i\varphi}z^2}. \quad (10)$$

Also belongs to the class S .It displays a range of E on the plane, cut with for two on the line rays, outgoing from a point $\frac{e^{-i\varphi}}{4\sin^2\frac{\lambda}{2}}$ and from point $-\frac{e^{-i\varphi}}{4\sin^2\frac{\lambda}{2}}$ off to infinity. Remark that the function $L(z,\varphi,0)=K(z,\varphi)$ and function $L(z,\varphi,\pi)=K_1(z,\varphi)$ display the circle E on the plane , cut with one by one beam.

Accordance with properties of the Schwarz derivative,

$$\{L(z,\varphi,\lambda),z\}=\{K(z,\varphi),z\}=-6e^{2i\varphi}.$$

For the function $\frac{1}{\rho}K(\rho z,\varphi)\in S$, $0\le\rho\le 1$, have a $J(0)=-\rho^2e^{2i\varphi}$. By these remarks completes the proof about coincidence $J(0)$ with closure of unit circle.

Theorem 2.

The set of values of the derivative Schwartz $\{f,z\}$ on the class S under $z_0=0$ It is a vicious circle of radius 6 with center at first. Every boundary point $e^{i\theta}$ in this set make function (10) at $\lambda\in[0,\pi]$ and $e^{2i\varphi}=-e^{i\theta}$.

Referring to the general case $z_0\in E\setminus\{0\}$, we use that if $f(z)\in S$, in the same class include function

$$g(z)=\frac{f\left(\frac{z+z_0}{1+\overline{z_0}z}\right)-f(z_0)}{f'(z_0).(1-|z_0|^2)}$$

Accordance with properties of the Schwarz derivative,

$$\{g,z\}=\left\{f\left(\frac{z+z_0}{1+\overline{z_0}z}\right),z\right\}=\{f(w),w\}\left(\left(\frac{z+z_0}{1+\overline{z_0}z}\right)'\right)^2=\{f(w),w\}\frac{\left(1-|z_0|^2\right)^2}{\left(1+\overline{z_0}z\right)^4}$$

, $w=\frac{z+z_0}{1+\overline{z_0}z}$

Assuming that the $z=0$ and applying Theorem 2, we obtain the inequality

$$|\{f(z_0),z_0\}|\le\frac{6}{\left(1-|z_0|^2\right)^2},$$

defining $\Delta(z_0)$. Point $-\frac{6}{\left(1-|z_0|^2\right)^2}e^{i\theta}$, is introduced functions

$$f(z,\varphi,\lambda,z_0)=\frac{L\left(\frac{z+z_0}{1+\overline{z_0}z},\varphi,\lambda\right)-L(z_0,\varphi,\lambda)}{L'(z_0,\varphi,\lambda)\left(1-|z_0|^2\right)}$$

Where $L(z,\varphi,\lambda)$ defined by the formula (10), $0\le\lambda\le\pi$ and $e^{2i\varphi}=-e^{i\theta}$, that is functions

$$f(z,\varphi,\lambda,z_0)=\left(1+|z_0|^2\right)\left(\int_0^z\frac{\left(1+\overline{z_0}u\right)^2-e^{2i\varphi}(u+z_0)^2}{\left(\left(1+\overline{z_0}u\right)^2-2\cos\lambda e^{i\varphi}(u+u_0)+e^{2i\varphi}(u+u_0)^2\right)^2}du-\right.$$

$$\left.-\int_0^{z_0}\frac{\left(1+\overline{z_0}u\right)^2-e^{2i\varphi}(u+z_0)^2}{\left(\left(1+\overline{z_0}u\right)^2-2\cos\lambda e^{i\varphi}(u+u_0)+e^{2i\varphi}(u+u_0)^2\right)^2}du\right).$$

From the method of construction of the function $f(z,\varphi,\lambda,z_0)$ follows that it follows that it displays a circle E on a plane cut with for two lying on the line rays, outgoing from a points

$$e^{-i\varphi}\frac{\left(1-2\cos\lambda z_0+z_0^2\right)\left(1-2\cos\lambda z_0+z_0^2-4\sin^2\frac{\lambda}{2}z_0\right)}{4\sin^2\frac{\lambda}{2}z_0\left(1-z_0^2\right)\left(1-|z_0|^2\right)}$$

And from point

$$-e^{-i\varphi}\frac{\left(1-2\cos\lambda z_0+z_0^2\right)\left(1-2\cos\lambda z_0+z_0^2-4\cos^2\frac{\lambda}{2}z_0\right)}{4\cos^2\frac{\lambda}{2}z_0\left(1-z_0^2\right)\left(1-|z_0|^2\right)}$$

Off to infinity. Thus, the theorem is proved.

Theorem 3.

Let $f(z)\in S$ and $z_0\in E\setminus\{0\}$. Then the set of values of the functional $\{f,z_0\}$ on the class S is defined by the inequality

$$|\{f,z_0\}|\le\frac{6}{\left(1-|z_0|^2\right)^2}.$$

Boundary point $-\frac{6}{\left(1-|z_0|^2\right)^2}e^{i\theta}$ realized only functions $f(z,\varphi,\lambda,z_0)$ under $\lambda\in[0,\pi]$ and $e^{2i\varphi}=-e^{i\theta}$.

Inequality for Schwartz derivative on class S_M

Establish inequality between the modules function, derivative and derivative Schwartz on the class S_M, $1<M<\infty$.

Theorem 4.

If $f\in S_M$ and $z_0\in E$, then

$$\left|\{f(z_0),z_0\}\right|+\frac{6M^2|f'(z_0)|^2}{\left(M^2-|f(z_0)|^2\right)^2}\le\frac{6}{\left(1-|z_0|^2\right)^2}. \qquad (11)$$

Function $w=a\frac{f(z)}{M}$, $|a|=1$, univalent conformal displays E to E with preservation of zero.

Let $F(w)=w+c_2w^2+...\in S$, $|w|=1$. Then

$$\Psi(z)=\bar{a}MF\left(\frac{af(z)}{M}\right)=f(z)+\frac{a}{M}f^2(z)+...\in S$$

.And Accordance with properties of the Schwarz derivative,

$$\{\Psi(z_0),z_0\}=\{F(w),w\}\left(\frac{af'(z_0)}{M}\right)^2+\{f(z_0),z_0\}$$

.Where $w=a\frac{f(z_0)}{M}$. Let us choose $a=e^{-i\arg f'(z_0)}$.

Then $af'(z_0)=|f'(z_0)|$ and

$$\{\Psi(z_0),z_0\}=\{F(w),w\}\frac{|f'(z_0)|^2}{M^2}+\{f(z_0),z_0\}$$

.Hence by virtue Theorem 3

$$\left|\{f(z_0),z_0\}+\{F(w),w\}\frac{|f'(z_0)|^2}{M^2}\right|\le\frac{6}{\left(1-|z_0|^2\right)^2}.$$

Now we take this function, which corresponds to be an arbitrary boundary point in the field functional values $\{F(w_0),w_0\}$ on the class S under

$w_0=e^{-i\arg f'(z_0)}\frac{f(z_0)}{M}$. We get inequality

$$\left|\{f(z_0),z_0\}+\frac{6M^2|f'(z_0)|e^{i\theta}}{\left(M^2-|f(z_0)|^2\right)^2}\right|\le\frac{6}{\left(1-|z_0|^2\right)^2}$$,under

that $\theta=\arg\{f(z_0),z_0\}$ gives (11).The sign of equality in (11) will take place only in the event that , if the function $\Psi(z)$ realizes a boundary point $-\frac{6}{\left(1-|z_0|^2\right)^2}e^{i\theta}$ the range of values $\{\Psi(z_0),z_0\}$ .It allows finding all functions $f\in S_M$, for which (11) It takes place The sign of equality. In the case where $z_0=0$, the boundary point $\left(1-\frac{1}{M^2}\right)e^{i\theta}$ is entered only function

$$f_M(z,\varphi,\lambda)=\frac{2z}{1-2\left(1-\frac{1}{M}\right)\cos\lambda e^{i\varphi}+e^{2i\varphi}z^2+\sqrt{\left(1-2\left(1-\frac{1}{M}\right)\cos\lambda e^{i\varphi}+e^{2i\varphi}z^2\right)^2-\frac{4}{M^2}e^{2i\varphi}z^2}}$$

With $\lambda\in[0,\pi]$ and $e^{2i\varphi}=-e^{i\theta}$ and It considered that continuous branch of the radical that becomes unity at $z=0$.From the method of construction of the function $f_M(z,\varphi,\lambda)$ it follows that it displays a circle E on the circle $U=\{w\in\mathbb{C}:|w|<M\}$, from which two segments from points removed

$$e^{-i\varphi}M^2\left(1-\left(1-\frac{1}{M}\right)\cos\lambda-\sqrt{\left(1-\left(1-\frac{1}{M}\right)\cos\lambda\right)^2-\frac{1}{M^2}}\right)$$

And

$$-e^{-i\varphi}M^2\left(1-\left(1+\frac{1}{M}\right)\cos\lambda-\sqrt{\left(1-\left(1+\frac{1}{M}\right)\cos\lambda\right)^2-\frac{1}{M^2}}\right)$$

According to the points $e^{-i\varphi}M$ and $-e^{-i\varphi}M$ In the general case , when $z_0\in E\setminus\{0\}$ the equality sign holds only for the functions

$$f(z)=\frac{\omega^2-|g(-z_0)|^2}{g(-z_0)\left(1-|z_0|^2\right)}\frac{g\left(\frac{z-z_0}{1-z_0z}\right)-g(-z_0)}{\omega^2-\overline{g(-z_0)}g\left(\frac{z-z_0}{1-\overline{z_0}z}\right)}$$

Where $g(z)=f_\omega\left(z,\varphi,\sigma(1-\frac{1}{\omega})\right)$, $1<\omega<\pi$,

$0\le\sigma\le\pi$, and ω – root of the equation

$$\frac{\omega^2-\left|f_\omega\left(-z,\varphi,\sigma(1-\frac{1}{\omega})\right)\right|^2}{\omega\left|f'_\omega\left(-z,\varphi,\sigma(1-\frac{1}{\omega})\right)\right|\left(1-|z_0|^2\right)}=M. \qquad (12)$$

In this case $\frac{M|f'(z_0)|}{M^2-|f(z_0)|^2}=\frac{1}{\omega}$ and

$$\{f(z_0),z_0\}+\frac{6}{\omega^2}=-\frac{6e^{i\theta}}{\left(1-|z_0|^2\right)^2},$$

With $e^{2i\varphi}=-e^{i\theta}$.

Equation (12) for any M, $1<M<\infty$, it has at least one solution.

Other methods investigated area Schwartzman values I.A. Aleksandrov [4], U.E. Alenicyn [5], N.A Lebedev [6].

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