Journal of Multidisciplinary Engineering Science Studies (JMESS)
ISSN: 2912-1309
Vol. 2 Issue 1, January - 2016

Integral mean value method for solving Fredholm integral equations of the second kind on the half line

Zakieh Avazzadeh
Institute of Mathematics, School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China
z.avazzadeh@yahoo.com

Mohammad Heydari
Department of Mathematics, Yazd University, Yazd, Iran
m.heydari85@gmail.com, m.heydari@yazd.ac.ir

***Abstract*—In this paper, we develop the integral mean value method (IMVM) for solving Fredholm integral equations of the second kind on the half line. The most important point of the IMVM is simplicity, accuracy and competency. However the mean value theorem is not defined for improper integrals, we will apply the integral mean value theorem indirectly to achieve the required linearly independent equations in IMVM. We solve some examples to illustrate the applicability and simplicity of the method. The numerical results show that the method is efficient and accurate.**

Keywords—component; Integral equations on the half line, Integral mean value theorem, Integral mean value method (IMVM) Improper integrals.

I. INTRODUCTION

Integral equations appear in different kind and describe various events in science and engineering. Many initial and boundary value problems associated with ordinary differential equation (ODE) and partial differential equation (PDE) can be transformed into various kinds of integral equations [1, 2, 3, 4]. Since the considerable amount of natural phenomena in the real world are related to the parameters defined on semi-infinite or infinite interval, we need to the efficient methods to solve the equations which are involved to the improper integrals. Integral equations on unbounded domain can be solved by not many numerical and analytical methods and the efforts to find the qualified methods continue as a case study. In this paper, we are interested in solving the following Fredholm integral equations of the second kind on the half line

$$u(x) = f(x) + \lambda \int_0^{\infty} k(x,t) G\big(u(t)\big) dt, \quad x,t \in [0,\infty), \tag{1}$$

where λ is a real number, also G, f and k are given continuous functions on $[0,\infty)$, and $u(x)$ is an unknown function to be determined. The improper integral in (1) is considered as

$$\lim_{b\to\infty} \int_0^b k(x,t) G(u(t)) dt, \tag{2}$$

and its value is assumed to exist. The various numerical methods have been proposed for solving the unbounded Fredholm integral equations which apply Galerkin method with Laguerre Polynomials [5], quadrature methods [6, 7], projection methods [8], interpolation method based on Hermite zeros [9], Nyström's method based on composite quadratures [10], Nyström's method based on a product quadrature with zeros of Laguerre polynomials [11], graded mesh methods [12] and many other methods [13, 14, 15, 16, 17]. This study is an effort to generalize the integral mean value method (IMVM) which is established by Avazzadeh et al. [18, 19] for solving linear and nonlinear Fredholm integral of the second kind on the half line. The main idea is deformation of improper integral to new proper integral on the bounded interval which will be prepared to applying the mean value theorem. According to the integral mean value method, the integral equation will be reduced to the system of algebraic equations. The obtained system can be solved by Newton's method or other well-known efficient methods.

This paper is organized as follows: Section 2 briefly reviews the description of the IMVM based on [18] for solving the Fredholm integral equations. In Section 3, we develop the IMVM to solve Fredholm integral equations on the half line. Some numerical experiments are given in Section 4 to show the efficiency of the proposed method. Finally, a brief conclusion is drawn in Section 5.

II. DESCRIPTION OF IMVM

Consider the Fredholm integral equation of the second kind as follows

$$u(x) = f(x) + \lambda \int_a^b k(x,t) G(u(t)) dt, \quad x,t \in [a,b], \tag{3}$$

where λ is a real number, also G, f and k are given continuous functions, and u is unknown function to be determined. Now we apply the integral mean value theorem for solving the above integral equation.

Mean value theorem for integrals [18,19]. If $\omega(x)$ is continuous on $[a,b]$, then there is a point $c \in [a,b]$ such that

$$\int_a^b \omega(x)dx = (b-a)\omega(c). \quad (4)$$

Now we describe how (3) can be solved using the integral mean value theorem. By applying (4) into (3) we can get

$$u(x) = f(x) + \lambda(b-a)k(x,c(x))G(\mathrm{u}(c(x))), \quad (5)$$

where $c(x) \in [a,b]$ and $x \in [a,b]$.

Obviously, the integral $\int_a^b k(x,t)G(u(t))dt$ depends on x and the number c generally must be dependent on the variable x. That is, the number c should be a function with respect to x and here we write it as $c(x)$. In practice, to be able to implement our algorithm, we take $c(x)$ as a constant [19]. This assumption results in

$$u(x) = f(x) + \lambda(b-a)k(x,c)G(u(c)), \quad (6)$$

where $c \in [a,b]$. Therefore, finding the value of c and $\mathrm{u}(c)$ lead to obtain the solution of integral equation. The way to find c and $u(c)$ is reported from [18] as the following algorithm

Algorithm

i) Substitute c into (6) which gives

$$u(c) = f(c) + \lambda(b-a)k(c,c)G(u(c)). \quad (7)$$

ii) Replace (6) into (3) which gives

$$u(x) = f(x) + \lambda \int_a^b k(x,t)G(f(t) + \lambda(b-a) k(t,c)G(u(c)))dt. \quad (8)$$

iii) Let c into (8) which leads to

$$u(c) = f(c) + \lambda \int_a^b k(c,t)G(f(t) + \lambda(b-a) k(t,c)G(u(c)))dt. \quad (9)$$

iv) Solve the obtained equations (7) and (9) simultaneously.

Consecutive substitutions provide the needful linearly independent equations. For solving the above nonlinear system, we can use the various methods [20]. Here, Newton's method is used to solve the obtained system.

III. IMVM FOR SOLVING FREDHOLM INTEGRAL EQUATIONS ON THE HALF LINE

Consider the following infinite integral equation of the second kind

$$u(x) = f(x) + \lambda \int_0^\infty k(x,t)G(u(t))dt, \quad x,t \in [0,\infty), \quad (10)$$

where λ is a real number, also G, f and k are given continuous functions on $[0,\infty)$. For solving the above equation, we apply the integral mean value theorem indirectly because the mean value theorem is not valid for unbounded intervals. Firstly, we change variables x and t to obtain an integral equation on $(0,1]$ as follows

$$x = -\ln s, \quad t = -\ln\tau, \quad x,t \in [0,\infty), \ s,\tau \in (0,1], \quad (11)$$

which gives

$$u(-\ln s) = f(-\ln s) - \lambda \int_0^1 \frac{1}{\tau} k(-\ln s, -\ln\tau)G(u(-\ln\tau))d\tau, \quad s,t \in (0,1]. \quad (12)$$

To easier notation, we rewrite the above equation as follows

$$U(s) = F(s) - \lambda \int_0^1 K(s,\tau)G(U(\tau))d\tau, \quad s,\tau \in (0,1], \quad (13)$$

where

$$U(s) = u(-\ln s), \quad F(s) = f(-\ln s),$$

$$K(s,\tau) = \frac{1}{\tau}k(-\ln s, -\ln\tau). \quad (14)$$

The function $K(s,\tau)G(U(\tau))$ is continuous on $(0,1]$, but as you know $(0,1]$ is not a compact interval and consequently we cannot use the mean value theorem. For overcoming this problem we assume that $\varepsilon > 0$ is a sufficiently number close to zero, then the equation (13) can be reformed as follows:

$$U(s) = F(s) - \lambda \int_0^\varepsilon K(s,\tau)G(U(\tau))d\tau - \lambda \int_\varepsilon^1 K(s,\tau)G(U(\tau))d\tau, \quad s,\tau \in (0,1]. \quad (15)$$

It is obvious that:

$$\lim_{\varepsilon \to 0^+} \int_0^\varepsilon K(s,\tau)G(U(\tau))d\tau = 0, \quad (16)$$

and then the relation (15) will be substituted as:

$$U(s) \simeq F(s) - \lambda \int_\varepsilon^1 K(s,\tau)G(U(\tau))d\tau, \quad \tau \in [\varepsilon,1]. \quad (17)$$

Now we implement the IMVM algorithm described in previous section step by step on (17). According to the method we will have c and $U(c)$ as the unknowns and two algebraic equations which can solve by Newton's method. Note that the constructed $U(s)$ based on the following form

$$U(s) \simeq F(s) - \lambda(1-\varepsilon)K(s,c)G(U(c)), \quad (18)$$

has to be transformed to get the approximated solution of (10). Then the following transformation gives the final approximated solution of integral equation

$$s = e^{-x}, \quad x \in [0,\infty) \quad and \quad s \in (0,1]. \quad (19)$$

The illustrative examples shown in the next section demonstrate more details of the proposed method in performance.

IV. NUMERICAL EXPERIMENTS

In this section, we implement some examples to illustrate the achievements by the propose method. We often consider only 10 significant digits and $\varepsilon = 10^{-8}$ for solving the following test problems. Also, the programming is performed by Maple 16. The numerical experiments shows the simplicity and accuracy of the method. Note that solving the integral equations on unbounded domain are naturally complicated by using the numerical methods. In addition, the analytical methods often impose some restrictions on kernel of the integral. It seems the presented method can be considered in the research involving the infinite integral equation.

Example 1. [5] Let $\lambda = 1$ and

$$G(x) = x, \quad k(x,t) = \mathrm{e}^{-t^2-x}, \quad f(x) = x^5 - \mathrm{e}^{-x}, \tag{20}$$

with the exact solution $u(x) = x^5$. Due to the (11) we reform the integral equation as follows

$$K(s,\tau) = \frac{1}{\tau}\,\mathrm{e}^{-(\ln\tau)^2+\ln s}, \qquad F(s) = -(\ln s)^5 - s.$$

Corresponding to (7) and (9) in the presented algorithm for integral equation (17), we have two algebraic equations as

$$U(c) + (\ln c)^5 + c - \frac{\mathrm{e}^{-(\ln c)^2+\ln c}\,U(c)}{c} = 0,$$

$$U(c) + (\ln c)^5 + c + \frac{1}{2}\Big(0.8498918381\, c\mathrm{e}^{\frac{1}{4}+(\ln c)^2} - 2\,c\,\mathrm{e}^{(\ln c)^2} - 1.091282722\,U(c)\Big)\mathrm{e}^{-(\ln c)^2} = 0.$$

Using the different well-known methods for solving the obtained system containing two unknowns c and $U(c)$ gives

$$U(c) = 1.000000001, \quad c = 0.3678794411.$$

Hence, according to (18) we have

$$U(s) \simeq -(\ln s)^5 - s + 2.718281832\,\mathrm{e}^{-1+\ln s},$$

which means

$$u(x) \simeq x^5 + 10^{-9} \times \mathrm{e}^{-x}. \tag{21}$$

The exact and approximated solutions and absolute error function are illustrated in Figure 1. We emphasize the number of significant digits for solving this problem was considered 10. The experiment for 20 digits gives the following answer

$$u(x) \simeq x^5 + 10^{-20} \times \mathrm{e}^{-x}. \tag{22}$$

Obviously, the obtained solution is semi-analytic and the accuracy of IMVM depends on the accuracy of applied method for solving the system of algebraic equations.

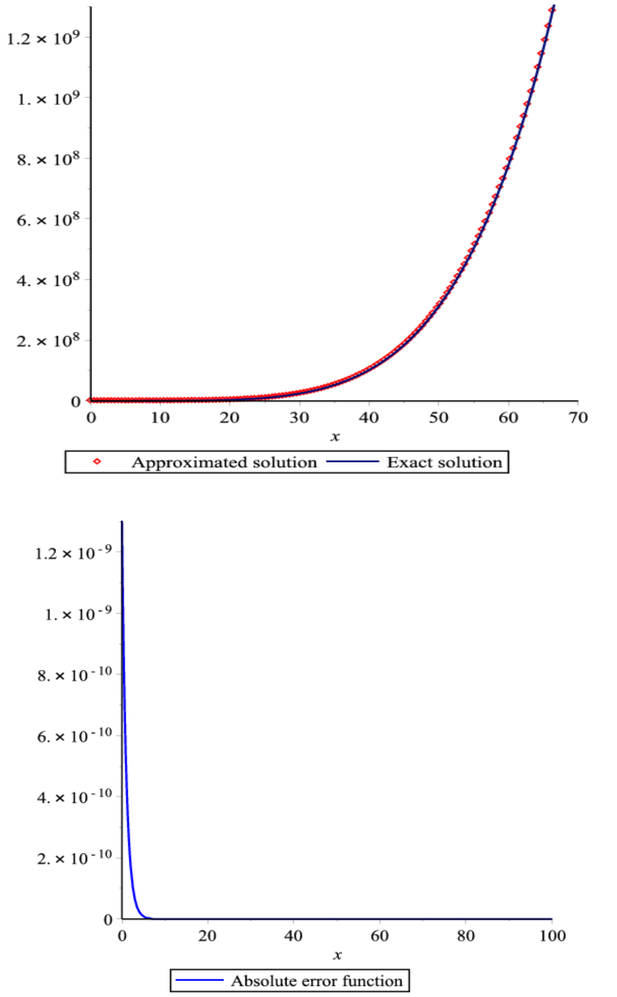

Figure 1: Exact and approximated solutions and absolute error function for Example 1.

Example 2. [5] Let $\lambda = 1$ and

$$G(x) = x, \quad k(x,t) = \mathrm{e}^{-t^2-x^2}, \quad f(x) = x^4 - \frac{3\sqrt{\pi}}{8}\mathrm{e}^{-x^2}, \tag{23}$$

with the exact solution $u(x) = x^4$. Similarly, we perform the algorithm to obtain the following results

$$U(c) = 0.5977326611, \quad c = 0.41508203951,$$

and the constructed solution based on (18) and (19) is as follows

$$u(x) \simeq x^4 - 10^{-10} \times \mathrm{e}^{-x^2}.$$

The exact and approximated solutions and absolute error function are shown in Figure 2.

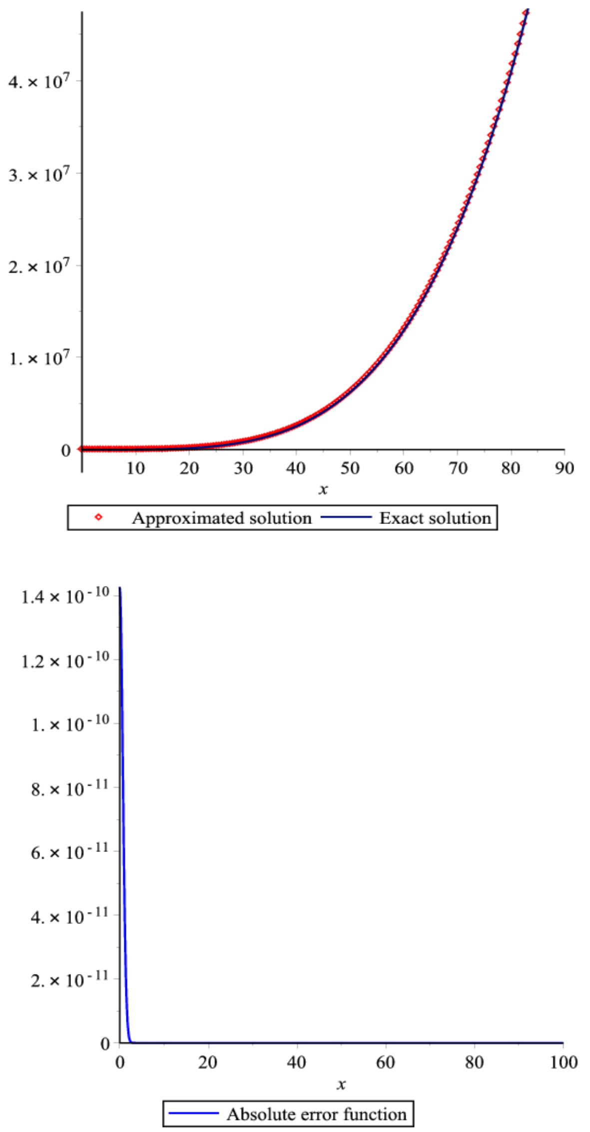

Figure 2: Exact and approximated solutions and absolute error function for Example 2.

Example 3. Let $\lambda = 1$ and

$$k(x,t) = \frac{1}{1+t^2}, \qquad G(x) = x^2, \tag{24}$$

where $f(x)$ is compatible with the exact solution $u(x) = x^4 e^{-x}$. Similarly, we obtain the following values

$$U(c) = 1.8672963718, \quad c = 0.1553793395.$$

So, the approximated solution based on (18) and (19) is as follows

$$u(x) \simeq x^4 e^{-x} + 3.36 \times 10^{-4},$$

with considering 64 significant digits. The exact and approximated solutions and absolute error function are shown in Figure 3.

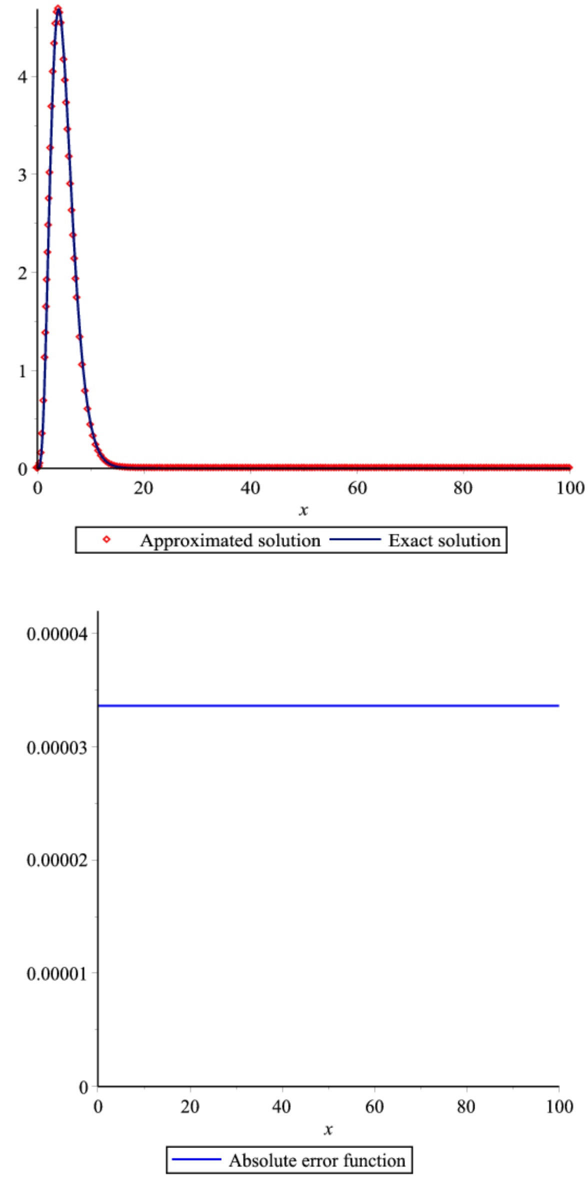

Figure 3: Exact and approximated solutions and absolute error function for Example 3.

Example 4. Let $\lambda = 1$ and

$$k(x,t) = \frac{1}{1+t^2}, \qquad G(x) = x, \tag{25}$$

where $f(x)$ is compatible with the exact solution $u(x) = e^{-x}\cos(x)$. The answer of obtained system of algebraic equations is as follows

$$U(c) = 0.3540268821, \qquad c = 0.4783125332,$$

with considering 64 significant digits. Thus the approximated solution based on (18) and (19) will be as follows

$$u(x) \simeq e^{-x}\cos(x) + 2.38 \times 10^{-13}.$$

The exact and approximated solutions and absolute error function are shown in Figure 4.

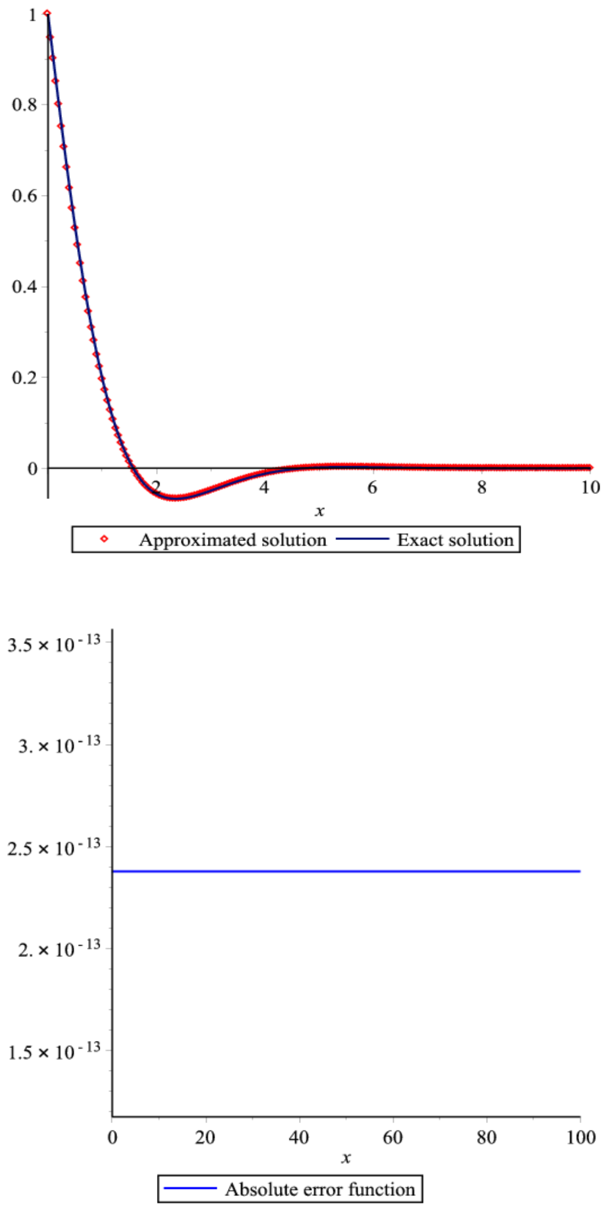

Figure 4: Exact and approximated solutions and absolute error function for Example 4.

Example 5. Let $\lambda = 1$ and

$$k(x,t) = \frac{1}{250+10\,(x^2+t^2)},$$

$$f(x) = \frac{1}{1+x^2} - \frac{1}{20}\,\frac{\pi\left(\sqrt{x^2+25}-1\right)}{(x^2+24)\sqrt{x^2+25}},$$

$$G(x) = x,$$

with the exact solution $u(x) = \frac{1}{1+x^2}$. The answer of obtained system of algebraic equations is as follows

$$U(c) = 0.1902059053, \qquad c = 0.1270816580,$$

with considering 10 significant digits. Thus the approximated solution based on (18) and (19) will be as follows

$$u(x) \simeq \frac{1}{1+x^2} - \frac{1}{20}\frac{\pi(\sqrt{x^2+25}-1)}{(x^2+24)\sqrt{x^2+25}} + \frac{1.496721937}{10x^2+292.5566130}.$$

The exact and approximated solutions and absolute error function are shown in Figure 5.

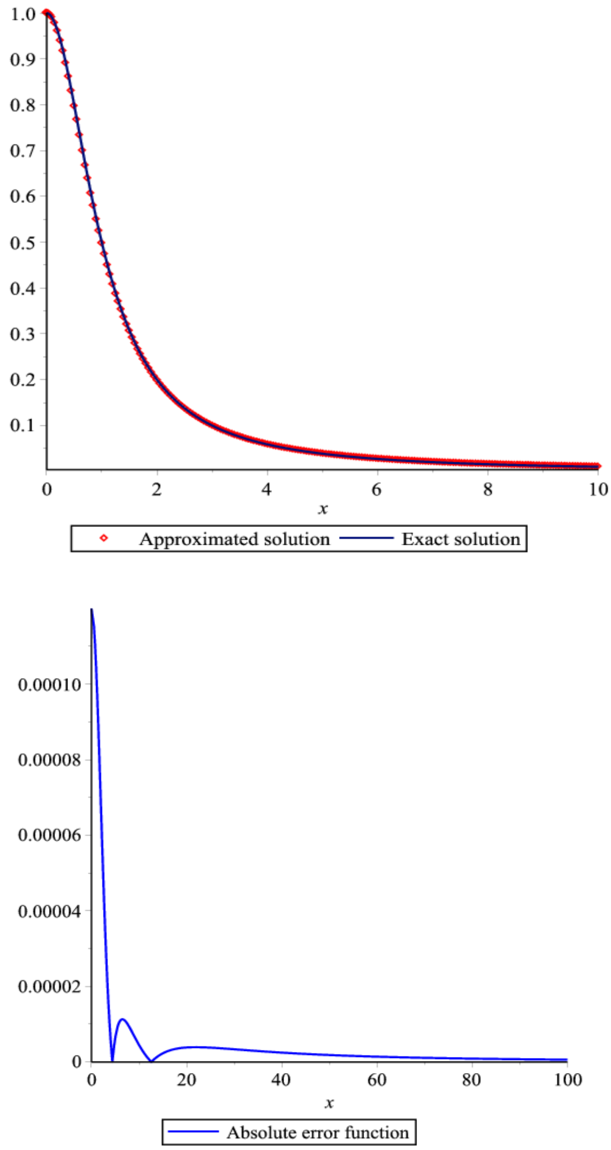

Figure 5: Exact and approximated solutions and absolute error function for Example 5.

V. Conclusion

The demonstrated examples in previous section imply that method is fast, simple and accurate with semi-analytical solution. The large amount of problem involving semi-infinite domain can be solved by the presented method. We have to emphasize the described transformation (11) is not only valid mapping from semi-infinite domain but also the problems defined on infinite domain $(-\infty, \infty)$ are candidate for this method with feasible transformation. Although there is an ambiguity about the end points of interval, the proposed method can not be waived as a powerful tool for solving some infinite integral equations. The clarification about the end points is the case study for further research which can avoid to unexpected behavior of approximated solution.

ACKNOWLEDGMENT *(Heading 5)*

Z. A. thanks "Natural Science Foundation of Jiangsu Province (Grant No. BK20150964)" and "A Project Funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions".

REFERENCES

[1] C.R. Wylie, L.C. Barrett, Advanced Engineering Mathematics, McGraw-Hill: New York, 1982.

[2] H.T.H. Piaggio, An Elementary Treatise on Differential Equations and their Applications, G. Bell and Sons, Ltd.: London, 1920.

[3] F. Mirzaee, S. Bimesl, A new Euler matrix method for solving systems of linear Volterra integral equations with variable coefficients, Journal of the Egyptian Mathematical Society, 2014, pp. 238-248.

[4] F. Mirzaee, S. Bimesl, Solving systems of high-order linear differential difference equations via Euler matrix method, Journal of the Egyptian Mathematical Society, Doi:10.1016/j.joems.2014.05.003.

[5] N.M.A. Nik Long, Z.K. Eshkuvatov, M. Yaghobifar, M. Hasan, Numerical solution of infinite boundary integral equation by using Galerkin method with Laguerre polynomials, Proc. World. Acad. Sci. Eng. Technol. 2008, pp. 951-955.

[6] I.H. Sloan, Quadrature methods for integral equations of the second kind over infinite intervals, Math. Comput. 1981, pp. 511-523.

[7] P.P.B. Eggermont, C.H. Lubich, Operational quadrature methods for Wiener-Hopf integral equations, Math. Comput. 1993, pp. 699-718.

[8] C. Frammartino, C. Laurita, G. Mastroianni, On the numerical solution of Fredholm integral equations on unbounded intervals, J. Comput. Appl. Math. 2003, pp. 355-378.

[9] M.C. De Bonis, C. Frammartino, G. Mastroianni, Numerical methods for some special Fredholm integral equations on the real line, J. Comput. Appl. Math. 2004, pp. 225-243.

[10] G.A. Chandler, I.G. Graham, The convergence of Nyström methods for Wiener-Hopf equations, Numer. Math. 1988, pp. 345-364.

[11] G. Mastroianni, G. Monegato, Nyström interpolants based on zeros of Laguerre polynomials for some Wiener-Hopf equations, IMA J. Numer. Anal. 1997, pp. 621-642.

[12] D. Liang, B. Zhang, Numerical analysis of graded mesh methods for a class of second kind integral equations on the real line, J. Math. Anal. Appl. 2004, pp. 482-502.

[13] P.M. Anselone, H. Sloan, Numerical solutions of integral equations on the half line, Numer. Math. 1987, pp. 599-614.

[14] D. O'Regan, Approximation of solutions of nonlinear operator equation on the half line, Comput. Math. Appl. 1998, pp. 65-77.

[15] A. Molabahrami, Integral mean value method for solving a general nonlinear Fredholm integro-differential equation under the mixed conditions, Commun. Numer. Anal. 2013, pp. 1-15.

[16] A. Molabahrami, An algorithm based on the regularization and integral mean value methods for the Fredholm integral equations of the first kind, Appl. Math. Modell. 2013, pp. 9634-9642.

[17] M. Almousa, A. Ismail, Optimal homotopy asymptotic method for solving the linear Fredholm integral equations of the first kind, 2013, http://dx.doi.org/10.1155/2013/278097.

[18] Z. Avazzadeh, M. Heydari, G.B. Loghmani, Numerical solution of Fredholm integral equations of the second kind by using integral mean value theorem, Appl. Math. Modell. 2011, pp. 2374-2383.

[19] M. Heydari, Z. Avazzadeh, H. Navvabpour, G. B. Loghmani, Numerical solution of Fredholm integral equations of the second kind by using integral mean value theorem II: High dimensions, Appl. Math. Modell. 2013, pp. 432-442.

[20] B. Kaltenbacher, A. Neubauer, O. Scherzer, Iterative Regularization Methods for Nonlinear Ill-Posed Problems, Walter de Gruyter, 2008.