Dimensionless Analysis of Axially Vibrating Elastically Supported Beams

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***Abstract*— In this study dimensionless free axial vibration analysis of an elastically supported beam is made. The supports are modeled by elastic translational springs. The frequency values for the first three vibration modes of the beam are obtained for various values of spring constants and presented in the tables. Three I-profile beams are chosen for the numerical analysis. The frequency values for the spring constants of zero and infinity are also compared, respectively, with the ones of free, fixed and cantilever beams and nearly the exact values are obtained with negligible error percentages.**

***Keywords*— free axial vibration; dimensionless analysis; elastic support**

I. INTRODUCTION

In practice, the representation of a beam by a discrete model is an idealized model; however, in fact, beams have continuously distributed mass and elasticity. Mostly, especially for the axially vibration, beams are modeled as continuous systems having infinite number of degreed of freedom [1-8].

In this study, the dimensionless equation of motion of an axially vibrating beam is obtained and the free vibration analysis of an elastically supported axially vibrating beam is made. The elastic springs against translation are used to model the supports. The dimensionless differential equation of motion of the axially vibrating beam is solved by separation of variables method [9] and the dimensionless displacement function is obtained. The dimensionless boundary conditions are written for the elastic supports. The natural frequencies for the first three modes are obtained for the various values of the spring constants. The results obtained for the spring constant value of zero are compared with the frequency values of free beam whereas the ones for the spring constant value of infinity are compared with the frequency values of fixed beam. In addition, the frequencies obtained for the left end spring constant value of infinity and the right end spring constant value of zero are compared with the frequency values of cantilever beam.

II. EQUATION OF MOTION FOR AN AXIALLY VIBRATING BEAM

A. *Dimensional Equation of Motion*

An axially vibrating beam, given in Fig. 1, with the distributed mass m, the length L, the modulus of elasticity E, the cross-section area A and the axial rigidity AE has a dimensional differential equation of motion for free vibration as [10]

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{m}{AE}\frac{\partial^2 u(x,t)}{\partial t^2} = 0 \quad (1)$$

where u(x,t) is the displacement function of the beam in terms of both displacement x and time t. Application of the separation of variables method to (1) as in the form of (2) is commonly used in vibration analysis of beams.

$$u(x,t) = X(x).T(t) = X(x).[A.\sin(\omega t) + B.\cos(\omega t)] \quad (2)$$

In (2), X(x) is the eigenfunction named as shape function, T(t) is time function, ω is the eigenvalue of the solution named as natural frequency and A, B are the integration constants.

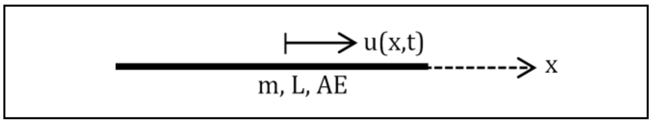

Fig. 1. *An axially vibrating beam with the distributed mass m, the length L, the modulus of elasticity E, the cross-section area A and the axial rigidity AE.*

The derivatives used in (1) can, therefore, be written as

$$\frac{\partial^2 u(x,t)}{\partial x^2} = u''(x,t) = X''(x).[A.\sin(\omega t) + B.\cos(\omega t)] = X''(x).T(t) \quad (3)$$

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \ddot{u}(x,t) = X(x).(-\omega^2)[A.\sin(\omega t) + B.\cos(\omega t)] = -\omega^2.X(x).T(t) \quad (4)$$

where ($''$) and ($\ddot{\ }$) denote the second order derivative due to x and t, respectively. Substitution of (3) and (4) in (1) gives the governing equation of motion in the dimensional form as

$$X''(x).T(t) + \frac{m\omega^2}{AE}X(x).T(t) = 0$$

$$X''(x) + \frac{m\omega^2}{AE}X(x) = 0 \quad 0 \le x \le L \quad (5)$$

B. *Dimensionless Equation of Motion*

Taking z=x/L as the dimensionless displacement variable with x=z.L, ∂x=∂z.L and $\partial x^2=\partial z^2.L^2$, the dimensionless differential equation of motion for axial vibration of a beam is obtained from (1) as

$$\frac{\partial^2 u(z,t)}{\partial z^2 L^2}-\frac{m}{AE}\frac{\partial^2 u(z,t)}{\partial t^2}=0 \qquad \frac{\partial^2 u(z,t)}{\partial z^2}-\frac{mL^2}{AE}\frac{\partial^2 u(z,t)}{\partial t^2} \tag{6}$$

Substituting the successive differentiations of the dimensionless displacement function u(z,t) in (7) obtained, again, by using the method of separation of variables into (6) give (8) for general solution of dimensionless equation of motion.

$$u(z,t)=Z(z).T(t) \tag{7}$$

$$\frac{\partial^2 u(z,t)}{\partial z^2}=u''(z,t)=Z''(z).T(t)$$

$$\frac{\partial^2 u(z,t)}{\partial t^2}=\ddot{u}(z,t)$$

$$=Z(z).(-\omega^2)[A.\sin(\omega t)+B.\cos(\omega t)]$$

$$=-\omega^2.Z(z).T(t)$$

$$Z''(z)+\frac{mL^2\omega^2}{AE}Z(z)=0$$

$$for\ \alpha^2=\frac{mL^2\omega^2}{AE} \qquad Z''(z)+\alpha^2 Z(z)=0 \tag{8}$$

The characteristic equation and the solution of (8) is given as follows as D being d/dz:

$$D^2+\alpha^2=0 \quad \rightarrow \quad D_{1,2}=\pm i\alpha \tag{9}$$

$$Z(z)=C_1.\sin(\alpha z)+C_2.\cos(\alpha z) \qquad 0\le z\le 1 \tag{10}$$

(10) gives the dimensionless shape function of the axially vibrating beam due to the dimensionless displacement variable, z. Therefore, from (7), the dimensionless displacement function of the axially vibrating beam has the form of (11).

$$u(z,t)=[C_1.\sin(\alpha z)+C_2.\cos(\alpha z)].T(t) \tag{11}$$

III. Boundary Conditions

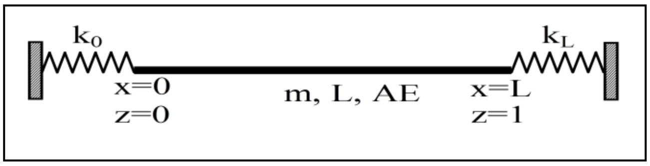

Fig. 2. *Elastically supported beam*

Two boundary conditions have to be written for the elastically supported beam in Fig. 2 since two integration constants (C_1, C_2) are obtained in the solution of second order differential equation of motion. The dimensional boundary conditions written for the left and the right ends of axially vibrating beam are given, respectively, as [11]

for x=0

$$N(x=0,t)=AEu'(x=0,t)=k_0.u(x=0,t) \tag{12}$$

for x=L

$$N(x=L,t)=AEu'(x=L,t)=-k_0.u(x=L,t) \tag{13}$$

where k_0 and k_L are the spring constant values of, respectively, the left end and the right end supports; N(z,t) being the axial force. Thus, the dimensionless boundary conditions for the same ends are obtained from (12) and (13) as in (14) and (15).

for z=0

$$N(z=0,t)=\frac{AE}{L}u'(z=0,t)=k_0.u(z=0,t) \tag{14}$$

for z=1

$$N(z=1,t)=\frac{AE}{L}u'(z=1,t)=-k_L.u(z=1,t) \tag{15}$$

If (11) and its derivative are substituted into (14) and (15) one gets the following relation between the coefficient matrix and the integration constants.

$$\begin{bmatrix}\propto & -k_{0D}\\ \propto.[k_{0D}+k_{LD}].\cos(\propto) & [k_{0D}.k_{LD}-\propto^2].\sin(\propto)\end{bmatrix}.\begin{Bmatrix}C_1\\C_2\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}$$

$$\begin{bmatrix}k_{11} & k_{12}\\ k_{21} & k_{22}\end{bmatrix}.\begin{Bmatrix}C_1\\C_2\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix} \quad\rightarrow\quad [k].\begin{Bmatrix}C_1\\C_2\end{Bmatrix}=\{0\} \tag{16}$$

$$|k|=\begin{vmatrix}k_{11} & k_{12}\\ k_{21} & k_{22}\end{vmatrix}=0 \tag{17}$$

where $[k]$ is the coefficient matrix, $k_{0D}=\frac{k_0.L}{AE}$ and $k_{LD}=\frac{k_L.L}{AE}$. For non-trivial solution equating the determinant of the coefficient matrix of (16) to zero, as in (17), will give the eigenfrequencies of the axially vibrating beam with elastic supports. These frequencies are computed by a program written by the author considering the secant method [12].

IV. Numerical Analysis

The first three natural frequencies of the axially vibrating beam with elastic supports are calculated for the dimensionless k_{0D} and k_{LD} values of 0, 10^1, 10^2, 10^3, ….., 10^9 and 10^{10}, the beam length of L=1 m. and the modulus of elasticity of E=2100000 kg/cm^2. IPB-100, IPB-300 and IPB-600 profiles are used for numerical analysis with the mechanical properties given in Table I where h is height, G is weight per length, A is cross-section area and AE is axial rigidity of the corresponding profile. The distributed mass of the beam m is calculated from G/g as g being the acceleration of gravity with the value of 981 cm/sn^2.

TABLE I. The Mechanical Properties of the Profiles Used in This Study

Profile	***h (cm)***	***G (kg/cm)***	***A (cm^2)***	***AE (kg)***
IPB100	10	0.081	10.3	21630000
IPB300	30	0.422	53.8	112980000
IPB600	60	1.22	156	327600000

The frequency values computed due to different values of the dimensionless spring constants for the both ends are presented in Tables II, III and IV for, respectively, IPB-100, IPB-300 and IPB-600.

TABLE II. FREQUENCIES OF IPB-100 OBTAINED FOR DIFFERENT VALUES OF DIMENSIONLESS SPRING CONSTANTS

k_{0D}=k_{LD}	ω_1	ω_2	ω_3
$\omega_i = n_i \frac{\pi}{L}\sqrt{\frac{AE}{m}}$	16079.4107	32158.8214	48237.2320
0	16079.4107	32158.8213	48237.2320
10^1	13450	27165	41290
10^2	15765	31530	47296
10^3	16048	32095	48142
10^4	16077	32153	48229
10^5	16079.0891	32158.1781	48237.2672
10^6	16079.3785	32158.7570	48237.1355
10^7	16079.4074	32158.8149	48237.2223
10^8	16079.4103	32158.8207	48237.2310
10^9	16079.4106	32158.8212	48237.2319
10^{10}	16079.4107	32158.8213	48237.2319
10^{11}	16079.4107	32158.8213	48237.2320
$\omega_i = n_i \frac{\pi}{L}\sqrt{\frac{AE}{m}}$	16079.4107	32158.8214	48237.2320

TABLE III. FREQUENCIES OF IPB-300 OBTAINED FOR DIFFERENT VALUES OF DIMENSIONLESS SPRING CONSTANTS

k_{0D}=k_{LD}	ω_1	ω_2	ω_3
$\omega_i = n_i \frac{\pi}{L}\sqrt{\frac{AE}{m}}$	16100.1134	32200.2269	48300.3403
0	16100.1134	32200.2269	48300.3403
10^1	13467	27200	41343
10^2	15785	31570	47356
10^3	16068	32136	48206
10^4	16097	32194	48291
10^5	16100	32200	48300
10^6	16100.0813	32200.1625	48300.2437
10^7	16100.1102	32200.2205	48300.3307
10^8	16100.1131	32200.2263	48300.3394
10^9	16100.1134	32200.2268	48300.3402
10^{10}	16100.1134	32200.2269	48300.3403
$\omega_i = n_i \frac{\pi}{L}\sqrt{\frac{AE}{m}}$	16100.1134	32200.2269	48300.3403

TABLE IV. FREQUENCIES OF IPB-600 OBTAINED FOR DIFFERENT VALUES OF DIMENSIONLESS SPRING CONSTANTS

k_{0D}=k_{LD}	ω_1	ω_2	ω_3
$\omega_i = n_i \frac{\pi}{L}\sqrt{\frac{AE}{m}}$	16124.1343	32248.2687	48372.4030
0	16124.1343	32248.2687	48372.4030
10^1	13487	27240	41405
10^2	15809	31617	47427
10^3	16092	32184	48276
10^4	16121	32242	48363
10^5	16124	32248	48372
10^6	16124.1021	32248.2042	48372.3063
10^7	16124.1311	32248.2622	48372.3933
10^8	16124.1340	32248.2680	48372.4020
10^9	16124.1343	32248.2686	48372.4029
10^{10}	16124.1343	32248.2687	48372.4030
$\omega_i = n_i \frac{\pi}{L}\sqrt{\frac{AE}{m}}$	16124.1343	32248.2687	48372.4030

The frequency values computed due to different values of the dimensionless spring constant k_0 and k_L=0 are presented in Tables V, VI and VII for, respectively, IPB-100, IPB-300 and IPB-600.

TABLE V. FREQUENCIES OF IPB-100 OBTAINED FOR DIFFERENT VALUES OF k_{0D} and k_{LD}=0

k_{LD}=0 and k_{0D}	ω_1	ω_2	ω_3
10^1	7314	22039	36996
10^2	7961	23881	39802
10^3	8032	24096	40159
10^4	8039	24117	40195
10^5	8039.6249	24118.8748	40198.1246

$k_{LD}=0$ and k_{0D}	ω_1	ω_2	ω_3
10^6	8039.6973	24119.0919	40198.4864
10^7	8039.7045	24119.1136	40198.5226
10^8	8039.7053	24119.1157	40198.5262
10^9	8039.7053	24119.1160	40198.5266
$\omega_i = \frac{(2n_i-1)}{2}\frac{\pi}{L}\sqrt{\frac{AE}{m}}$	8039.7053	24119.1160	40198.5266

TABLE VI. FREQUENCIES OF IPB-300 OBTAINED FOR DIFFERENT VALUES OF k_{0D} and $k_{LD}=0$

$k_{LD}=0$ and k_{0D}	ω_1	ω_2	ω_3
10^1	7323	22067	37043
10^2	7971	23912	39853
10^3	8043	24127	40211
10^4	8050	24148	40247
10^5	8050	24150	40250
10^6	8050.0487	24150.1460	40250.2434
10^7	8050.0559	24150.1678	40250.2796
10^8	8050.0567	24150.1699	40250.2832
10^9	8050.0567	24150.1701	40250.2836
$\omega_i = \frac{(2n_i-1)}{2}\frac{\pi}{L}\sqrt{\frac{AE}{m}}$	8050.0567	24150.1702	40250.2836

TABLE VII. FREQUENCIES OF IPB-600 OBTAINED FOR DIFFERENT VALUES OF k_{0D} and $k_{LD}=0$

$k_{LD}=0$ and k_{0D}	ω_1	ω_2	ω_3
10^1	7334	22100	37099
10^2	7983	23947	39913
10^3	8055	24163	40271
10^4	8062	24184	40307
10^5	8062	24186	40310
10^6	8062.0591	24186.1773	40310.2955
10^7	8062.0664	24186.1991	40310.3318
10^8	8062.0671	24186.2013	40310.3354
10^9	8062.0672	24186.2015	40310.3358
$\omega_i = \frac{(2n_i-1)}{2}\frac{\pi}{L}\sqrt{\frac{AE}{m}}$	8062.0672	24186.2015	40310.3358

V. CONCLUSIONS

In this study free longitudinal vibration of an elastically supported beam is made using dimensionless equation of motion and dimensionless boundary conditions. The natural frequency values are obtained for different values of spring constants at both ends and presented in tables. It can be seen from Tables 2, 3, 4 and Tables 6, 7 and 8 that as the spring constant values increase through a value of 10^5 the frequency values rapidly increase, however, at the value of 10^5 the frequency values are so close to its ideal limit obtained from the frequency equation of ideal support condition, being free or fixed. As the spring constant value increases from 10^5 to theoretically infinity (practically 10^{10} for both ends elastically supported beams and 10^9 for the beams with $k_{LD}=0$ in this study) the frequency values show gentle increase and at the spring constant value that represents infinity the frequency value reaches its limit value for the considering support. Increasing the height of the beam section causes also an increase in frequency values for all conditions considered in this study.

The frequencies at the first and the last rows of Tables II, III and IV are obtained, respectively, from the frequency equations of free and fixed beams which have the same frequency equation; and it is seen that the same frequency values are obtained for, respectively, $k_{0D}=k_{LD}=0$ and $k_{0D}=k_{LD}=10^{10}$ for all I-profiles.

The frequencies at the last rows of Tables V, VI and VII are obtained, from the frequency equation of cantilever beam; and it is seen that the same frequency values are obtained for, respectively, $k_{LD}=0$ and $k_{0D}=10^9$ for all I-profiles.

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