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Dimensionless Free Vibration Analysis of Axially Vibrating Beams with Tip Masses

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***Abstract*— In this study dimensionless free axial vibration analysis of beam with tip masses is made. The tip masses are modeled by concentrated masses at both ends of the beam. The frequency values for the first three vibration modes of the beam are obtained for various values of concentrated masses and presented in the tables. Three I-profile beams are chosen for the numerical analysis. The frequency values for the concentrated masses of zero and infinity are also compared, respectively, with the ones of free, fixed and cantilever beams and nearly the exact values are obtained with negligible error percentages.**

Keywords*— *free axial vibration; dimensionless analysis; tip mass

I. INTRODUCTION

In practice, the representation of a beam by a discrete model is an idealized model; however, in fact, beams have continuously distributed mass and elasticity. Mostly, especially for the axially vibration, beams are modeled as continuous systems having infinite number of degreed of freedom [1-8].

In this study, the dimensionless equation of motion of an axially vibrating beam is obtained and the free vibration analysis of an axially vibrating beam with tip masses is made. Two concentrated masses are used at both ends of the beam to model the tip masses. The dimensionless differential equation of motion of the axially vibrating beam is solved by separation of variables method [9] and the dimensionless displacement function is obtained. The dimensionless boundary conditions are written for the tip masses at both ends. The natural frequencies for the first three modes are obtained for the various values of the concentrated masses. The results obtained for the concentrated mass value of zero at both ends are compared with the frequency values of free beam whereas the ones for the concentrated mass value of infinity at both ends are compared with the frequency values of fixed beam. In addition, the frequencies obtained for the left end concentrated mass value of infinity and the right end concentrated mass value of zero are compared with the frequency values of cantilever beam.

II. EQUATION OF MOTION FOR AN AXIALLY VIBRATING BEAM

A. Dimensional Equation of Motion

An axially vibrating beam, given in Fig. 1, with the distributed mass m, the length L, the modulus of elasticity E, the cross-section area A and the axial rigidity AE has a dimensional differential equation of motion for free vibration as [10]

$$\frac{\partial^2 \mathrm{u(x,t)}}{\partial x^2} - \frac{m}{\mathrm{AE}}\frac{\partial^2 \mathrm{u(x,t)}}{\partial t^2} = 0 \qquad (1)$$

where u(x,t) is the displacement function of the beam in terms of both displacement x and time t. Application of the separation of variables method to (1) as in the form of (2) is commonly used in vibration analysis of beams.

$$u(x,t) = X(x).T(t) = X(x).[A.\sin(\omega t) + B.\cos(\omega t)] \quad (2)$$

In (2), X(x) is the eigenfunction named as shape function, T(t) is time function, ω is the eigenvalue of the solution named as natural frequency and A, B are the integration constants.

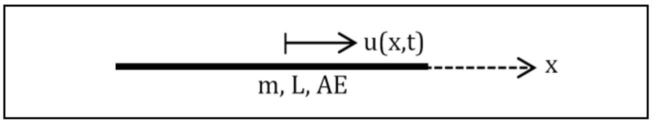

Fig. 1. *An axially vibrating beam with the distributed mass m, the length L, the modulus of elasticity E, the cross-section area A and the axial rigidity AE.*

The derivatives used in (1) can, therefore, be written as

$$\frac{\partial^2 u(x,t)}{\partial x^2} = u''(x,t) = X''(x).[A.\sin(\omega t) + B.\cos(\omega t)] = X''(x).T(t) \qquad (3)$$

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \ddot{u}(x,t) = X(x).(-\omega^2)[A.\sin(\omega t) + B.\cos(\omega t)] = -\omega^2.X(x).T(t) \qquad (4)$$

where ($^{//}$) and ($\ddot{}$) denote the second order derivative due to x and t, respectively. Substitution of (3) and (4) in (1) gives the governing equation of motion in the dimensional form as

$$X''(x).T(t) + \frac{m\omega^2}{\mathrm{AE}}X(x).T(t) = 0$$

$$X''(x) + \frac{m\omega^2}{\mathrm{AE}}X(x) = 0 \qquad 0 \le x \le L \qquad (5)$$

B. Dimensionless Equation of Motion

Taking z=x/L as the dimensionless displacement variable with x=z.L, ∂x=∂z.L and $\partial x^2=\partial z^2.L^2$, the dimensionless differential equation of motion for axial vibration of a beam is obtained from (1) as

$$\frac{\partial^2 \mathrm{u(z,t)}}{\partial z^2 L^2} - \frac{m}{\mathrm{AE}}\frac{\partial^2 \mathrm{u(z,t)}}{\partial t^2} = 0 \qquad \frac{\partial^2 \mathrm{u(z,t)}}{\partial z^2} - \frac{mL^2}{\mathrm{AE}}\frac{\partial^2 \mathrm{u(z,t)}}{\partial t^2} \qquad (6)$$

Substituting the successive differentiations of the dimensionless displacement function u(z,t) in (7) obtained, again, by using the method of separation of variables into (6) give (8) for general solution of dimensionless equation of motion.

$$u(z,t) = Z(z).T(t) \qquad (7)$$

$$\frac{\partial^2 u(z,t)}{\partial z^2} = u''(z,t) = Z''(z).T(t)$$

$$\frac{\partial^2 u(z,t)}{\partial t^2} = \ddot{u}(z,t)$$

$$= Z(z).(-\omega^2)[A.\sin(\omega t) + B.\cos(\omega t)]$$
$$= -\omega^2.Z(z).T(t)$$

$$Z''(z) + \frac{mL^2\omega^2}{\mathrm{AE}}Z(z) = 0$$

$$for\ \alpha^2 = \frac{mL^2\omega^2}{\mathrm{AE}} \qquad Z''(z) + \alpha^2 Z(z) = 0 \qquad (8)$$

The characteristic equation and the solution of (8) is given as follows as D being d/dz:

$$D^2 + \alpha^2 = 0 \quad \rightarrow \quad D_{1,2} = \pm i\alpha \qquad (9)$$

$$Z(z) = C_1.\sin(\alpha z) + C_2.\cos(\alpha z) \qquad 0 \le z \le 1 \qquad (10)$$

(10) gives the dimensionless shape function of the axially vibrating beam due to the dimensionless displacement variable, z. Therefore, from (7), the dimensionless displacement function of the axially vibrating beam has the form of (11).

$$u(z,t) = [C_1.\sin(\alpha z) + C_2.\cos(\alpha z)].T(t) \qquad (11)$$

III. Boundary Conditions

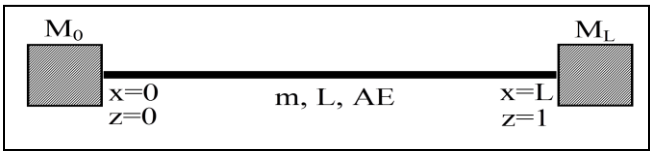

Fig. 2. Axially vibrating *beam with tip masses*

Two boundary conditions have to be written for the concentrated masses at both ends of the beam in Fig. 2 since two integration constants (C_1, C_2) are obtained in the solution of second order differential equation of motion. The dimensional boundary conditions written for the left and the right ends of axially vibrating beam with two concentrated masses are given, respectively, as [11]

for x=0

$$N(x=0,t) = AEu'(x=0,t) = M_0.\ddot{u}(x=0,t) \qquad (12)$$

for x=L

$$N(x=L,t) = AEu'(x=L,t) = -M_L.\ddot{u}(x=L,t) \qquad (13)$$

where M_0 and M_L are the concentrated mass values of, respectively, the left end and the right end supports; N(z,t) being the axial force. Thus, the dimensionless boundary conditions for the same ends are obtained from (12) and (13) as in (14) and (15).

for z=0

$$\mathrm{N}(z=0,t) = \frac{\mathrm{AE}}{L}\mathrm{u}'(z=0,t) = \mathrm{M_0}.\ddot{\mathrm{u}}(z=0,t) \qquad (14)$$

for z=1

$$\mathrm{N}(z=1,t) = \frac{\mathrm{AE}}{L}\mathrm{u}'(z=1,t) = -\mathrm{M_L}.\ddot{\mathrm{u}}(z=1,t) \qquad (15)$$

If (11) and its derivative are substituted into (14) and (15) one gets the following relation between the coefficient matrix and the integration constants.

$$\begin{bmatrix} \propto & M_{0D}.\omega^2 \\ [\propto.\cos(\propto) - M_{LD}.\omega^2.\sin(\propto)] & [-\propto.\sin(\propto) - M_{LD}.\omega^2.\cos(\propto)] \end{bmatrix}$$

$$.\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}.\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \rightarrow \quad [k].\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \{0\} \qquad (16)$$

$$|k| = \begin{vmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{vmatrix} = 0 \qquad (17)$$

where $[k]$ is the coefficient matrix, $M_{0D} = \frac{M_0.L}{AE}$ and $M_{LD} = \frac{M_L.L}{AE}$. For non-trivial solution equating the determinant of the coefficient matrix of (16) to zero, as in (17), will give the eigenfrequencies of the axially vibrating beam with tip masses. These frequencies are computed by a program written by the author considering the secant method [12].

IV. Numerical Analysis

The first three natural frequencies of the axially vibrating beam with concentrated masses at both ends are calculated for the dimensionless M_{0D} and M_{LD} values of 10^{-7}, 10^{-6}, 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 10^{0} and 10^{1}, the beam length of L=1 m. and the modulus of elasticity of E=2100000 kg/cm^2. IPB-100, IPB-300 and IPB-600 profiles are used for numerical analysis with the mechanical properties given in Table I where h is height, G is weight per length, A is cross-section area and AE is axial rigidity of the corresponding profile. The distributed mass of the beam m is calculated from G/g as g being the acceleration of gravity with the value of 981 cm/sn^2.

TABLE I. The Mechanical Properties of the Profiles Used in This Study

Profile	***h (cm)***	***G (kg/cm)***	***A (cm^2)***	***AE (kg)***
IPB100	10	0.081	10.3	21630000
IPB300	30	0.422	53.8	112980000

Profile	h (cm)	G (kg/cm)	A (cm^2)	AE (kg)
IPB600	60	1.22	156	327600000

The frequency values computed due to different values of the dimensionless concentrated masses for the both ends are presented in Tables II, III and IV for, respectively, IPB-100, IPB-300 and IPB-600.

TABLE II. FREQUENCIES OF IPB-100 OBTAINED FOR DIFFERENT VALUES OF DIMENSIONLESS CONCENTRATED MASSES

$M_{0D}=M_{LD}$	ω_1	ω_2	ω_3
$\omega_i = n_i \frac{\pi}{L}\sqrt{\frac{AE}{m}}$	16079.4107	32158.8214	48237.2320
0	16079.4107	32158.8213	48237.2320
10^{-7}	17235	32769	48650
10^{-6}	16203	32221	48280
10^{-5}	16092	32166	48243
10^{-4}	16081	32160	48239
10^{-3}	16079.5350	32158.8835	48238.2734
10^{-2}	16079.4231	32158.8275	48238.2361
10^{-1}	16079.4119	32158.8219	48238.2324
10^{0}	16079.4108	32158.8214	48238.2320
10^{1}	16079.4107	32158.8213	48237.2320
$\omega_i = n_i \frac{\pi}{L}\sqrt{\frac{AE}{m}}$	16079.4107	32158.8214	48237.2320

TABLE III. FREQUENCIES OF IPB-300 OBTAINED FOR DIFFERENT VALUES OF DIMENSIONLESS CONCENTRATED MASSES

$M_{0D}=M_{LD}$	ω_1	ω_2	ω_3
$\omega_i = n_i \frac{\pi}{L}\sqrt{\frac{AE}{m}}$	16100.1134	32200.2269	48300.3403
0	16100.1135	32200.2269	48300.3403
10^{-7}	17255	32810	48711
10^{-6}	16224	32263	48342
10^{-5}	16113	32207	48305
10^{-4}	16102	32201	48301
10^{-3}	16100.2377	32200.2890	48300.3817
10^{-2}	16100.1259	32200.2331	48300.3445
10^{-1}	16100.1147	32200.2275	48300.3408
10^{0}	16100.1136	32200.2270	48300.3404
10^{1}	16100.1135	32200.2269	48300.3403
$\omega_i = n_i \frac{\pi}{L}\sqrt{\frac{AE}{m}}$	16100.1134	32200.2269	48300.3403

TABLE IV. FREQUENCIES OF IPB-600 OBTAINED FOR DIFFERENT VALUES OF DIMENSIONLESS CONCENTRATED MASSES

$M_{0D}=M_{LD}$	ω_1	ω_2	ω_3
$\omega_i = n_i \frac{\pi}{L}\sqrt{\frac{AE}{m}}$	16124.1343	32248.2687	48372.4030
0	16124.1344	32248.2687	48372.4030
10^{-7}	17277	32857	48783
10^{-6}	16248	32311	48414
10^{-5}	16137	32255	48377
10^{-4}	16126	32249	48373
10^{-3}	16124.2584	32248.3307	48372.4443
10^{-2}	16124.1467	32248.2749	48372.4071
10^{-1}	16124.1356	32248.2693	48372.4034
10^{0}	16124.1345	32248.2687	48372.4030
10^{1}	16124.1344	32248.2687	48372.4030
$\omega_i = n_i \frac{\pi}{L}\sqrt{\frac{AE}{m}}$	16124.1343	32248.2687	48372.4030

The frequency values computed due to different values of the dimensionless concentrated mass M_{0D} and M_{LD}=0 are presented in Tables V, VI and VII for, respectively, IPB-100, IPB-300 and IPB-600.

TABLE V. FREQUENCIES OF IPB-100 OBTAINED FOR DIFFERENT VALUES OF M_{0D} and M_{LD}=0

M_{LD}=0 and M_{0D}	ω_1	ω_2	ω_3
10^{-7}	9120	24526	40446
10^{-6}	8163	24161	40224
10^{-5}	8053	24124	40202
10^{-4}	8041	24120	40199
10^{-3}	8039.8297	24119.1574	40198.5515
10^{-2}	8039.7178	24119.1201	40198.5291
10^{-1}	8039.7066	24119.1164	40198.5269

$M_{LD}=0$ and M_{0D}	ω_1	ω_2	ω_3
10^{0}	8039.7055	24119.1160	40198.5267
10^{1}	8039.7053	24119.1160	40198.5266
$\omega_i = \frac{(2n_i-1)}{2}\frac{\pi}{L}\sqrt{\frac{AE}{m}}$	8039.7053	24119.1160	40198.5266

TABLE VI. FREQUENCIES OF IPB-300 OBTAINED FOR DIFFERENT VALUES OF M_{0D} and $M_{LD}=0$

$M_{LD}=0$ and M_{0D}	ω_1	ω_2	ω_3
10^{-7}	9130	24557	40498
10^{-6}	8173	24192	40276
10^{-5}	8063	24155	40253
10^{-4}	8052	24151	40251
10^{-3}	8050.1810	24150.2116	40250.3085
10^{-2}	8050.0692	24150.1743	40250.2861
10^{-1}	8050.0580	24150.1706	40250.2839
10^{0}	8050.0569	24150.1702	40250.2836
10^{1}	8050.0567	24150.1702	40250.2836
$\omega_i = \frac{(2n_i-1)}{2}\frac{\pi}{L}\sqrt{\frac{AE}{m}}$	8050.0567	24150.1702	40250.2836

TABLE VII. FREQUENCIES OF IPB-600 OBTAINED FOR DIFFERENT VALUES OF M_{0D} and $M_{LD}=0$

$M_{LD}=0$ and M_{0D}	ω_1	ω_2	ω_3
10^{-7}	9141	24592	40557
10^{-6}	8185	24228	40336
10^{-5}	8075	24191	40313
10^{-4}	8064	24187	40311
10^{-3}	8062.1912	24186.2429	40310.3606
10^{-2}	8062.0796	24186.2056	40310.3383
10^{-1}	8062.0684	24186.2019	40310.3361
10^{0}	8062.0673	24186.2015	40310.3359
10^{1}	8062.0672	24186.2015	40310.3358
$\omega_i = \frac{(2n_i-1)}{2}\frac{\pi}{L}\sqrt{\frac{AE}{m}}$	8062.0672	24186.2015	40310.3358

V. CONCLUSIONS

In this study free longitudinal vibration of a beam carrying two concentrated masses at both ends is made using dimensionless equation of motion and dimensionless boundary conditions. The natural frequency values are obtained for different values of concentrated masses at both ends and presented in tables. It can be seen from Tables 2, 3, 4 and Tables 6, 7 and 8 that as the concentrated mass values increase through a value of 10^{-4} the frequency values rapidly decrease, however, at the value of 10^{-4} the frequency values are so close to its ideal limit obtained from the frequency equation of ideal support condition, being free, fixed or cantilever. As the concentrated mass value increases from 10^{-4} to theoretically infinity (practically 10^{1} in this study) the frequency values show gentle decrease and at the concentrated mass value that represents infinity the frequency value reaches its limit value for the considering support. Increasing the height of the beam section causes also an increase in frequency values for all conditions considered in this study.

The frequencies at the first and the last rows of Tables II, III and IV are obtained, respectively, from the frequency equations of free and fixed beams which have the same frequency equation; and it is seen that the same frequency values are obtained for, respectively, $M_{0D}=M_{LD}=0$ and $M_{0D}=M_{LD}=10^{1}$ for all I-profiles.

The frequencies at the last rows of Tables V, VI and VII are obtained, from the frequency equation of cantilever beam; and it is seen that the same frequency values are obtained for, respectively, $M_{LD}=0$ and $M_{0D}=10^{1}$ for all I-profiles.

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